On the center of quantized enveloping algebras

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Abstract

Let U be a quasitriangular Hopf algebra. One may use the R-matrix of U in order to construct scalar invariants of knots. Analogously, Reshetikhin wrote tangle invariants which take their values in the center of U. Reshetikhin's expressions thus define central elements in U. We prove here an identity caracterizing some of these elements, when U is a quantized enveloping algebra. As an application, we give a proof for a statement of Faddeev, Reshetikhin and Takhtadzhyan concerning the center of a quantized enveloping algebra.

Introduction

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and $\mathrm{U}_q\mathfrak{g}$ be the associated quantized enveloping algebra, at a generic value q of the parameter. There are at least three descriptions of the center $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$ of $\mathrm{U}_q\mathfrak{g}$. Rosso [15] defined a quantum analogue of the Harish-Chandra map in order to get an isomorphism between $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$ and the algebra of exponential invariants. Joseph and Letzter [8] described all the finite dimensional submodules of the adjoint module $\mathrm{U}_q\mathfrak{g}$: the center appears here as the isotypical component of the trivial type. Finally, Drinfel′d [4] used the universal R-matrix of $\mathrm{U}_q\mathfrak{g}$ in order to get a morphism from the representation ring of $\mathfrak{g}$ to $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$. Each of these constructions has its own advantage, and the links between them are known.

The construction of Drinfel′d uses only the universal R-matrix of $\mathrm{U}_q\mathfrak{g}$ and the quantum traces in $\mathrm{U}_q\mathfrak{g}$-modules: it is thus valid for the so-called "ribbon Hopf algebras". For these algebras, Reshetikhin [12] explained how invariants of certain tangles give rise to central elements. In this article, we give a formula connecting, in the case of $\mathrm{U}_q\mathfrak{g}$, some of these elements to the previous descriptions.

Let us go back to general results about the center of $\mathrm{U}_q\mathfrak{g}$. Faddeev, Reshetikhin and Takhtadzhyan [6] have exhibited a system of generators for $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$, but did not publish the proof of their theorem. Our result enables us to provide a proof for it, and even to give a more precise statement.

We will spend the first six sections to recall known constructions and results. New results appear in sections 2.4 and 3.

Acknowledgements: The author thanks Professors M. Rosso and O. Mathieu for numerous discussions on this subject or another, for encouragements and kind advice. He is supported by a grant from the french ministry of national education (bourse de doctorat).

1 Ribbons and central constructions

1.1 Ribbon Hopf algebras

Let U be a Hopf algebra over a field $\mathbb{K}$. The coproduct of an element $x \in \mathrm{U}$ will be written using Sweedler's notation: $\Delta_{\mathrm{U}}(x) = \sum x_{(1)} \otimes x_{(2)}$, the sum sign being generally omitted. We shall denote by ε_{U} and S_{U} the augmentation and the antipode of U. The context will make clear what Hopf algebra U is considered, so we will simply write Δ, ε and S.

We suppose now that U is quasitriangular for an R-matrix $R_{12} = \sum \alpha_i \otimes \beta_i \in \mathrm{U} \otimes \mathrm{U}$ (see [4] for the definition), and we put $R_{21} = \sum \beta_i \otimes \alpha_i \in \mathrm{U} \otimes \mathrm{U}$, $u = \sum \mathrm{S}(\beta_i)\alpha_i$. It is known that $(\mathrm{S} \otimes \mathrm{S})(R_{12}) = R_{12}$, that u is invertible in U, that $(\mathrm{U} \to \mathrm{U}, x \mapsto uxu^{-1})$ is the square of the antipode in U, and that $\Delta(u) = (R_{21}R_{12})^{-1}(u \otimes u)$.

A ribbon Hopf algebra (cf. [12]) is a quasitriangular Hopf algebra (U, R_{12}) given with a central element $v \in \mathrm{U}$ such that $v^2 = u\,\mathrm{S}(u)$, $v = \mathrm{S}(v)$, and uv^{-1} is group-like, so that:

$$\Delta(v) = (R_{21}R_{12})^{-1}(v \otimes v). \tag{1}$$

When U is infinite dimensional and R_{12} is an infinite sum, all these constructions are still valid in the framework of co-quasitriangular algebras (see [11, §10.2]). Let $\mathcal{A}$ be the restricted (Hopf) dual of U: then R_{12} may be viewed as a pairing $\mathcal{A} \times \mathcal{A} \to \mathbb{K}$, and u and v may be viewed as linear forms on $\mathcal{A}$. The fact that R_{12} intertwines the coproduct with its opposite is then written:

$$\forall f, g \in \mathcal{A}, \quad \langle f_{(1)} \otimes g_{(1)}, R_{12} \rangle f_{(2)} g_{(2)} = g_{(1)} f_{(1)} \langle f_{(2)} \otimes g_{(2)}, R_{12} \rangle.$$

It is not necessary for $\mathcal{A}$ to be the whole dual of U, but we will require that the canonical duality $\langle \cdot, \cdot \rangle$ between U and $\mathcal{A}$ is non-degenerate. In the following, we will use this set-up and these notations in order to deal with ribbon Hopf algebras.

1.2 Drinfel′d's construction of central elements

Let $(\mathrm{U}, \mathcal{A}, R_{12}, v)$ be a ribbon Hopf algebra. We endow U with the structure of a right U-module by the adjoint action: $x \cdot y = \mathrm{S}(y_{(1)}) x y_{(2)}$. We endow $\mathcal{A}$ with the structure of a right U-module by the coadjoint action: $\varphi \cdot y = \varphi(y_{(1)} \text{—} \mathrm{S}(y_{(2)})) = \langle y, \varphi_{(1)} \mathrm{S}(\varphi_{(3)}) \rangle \varphi_{(2)}$. We define the following pairing: $(\mathcal{A} \times \mathcal{A} \to \mathbb{K}, (\varphi, \psi) \mapsto \langle\langle \varphi, \psi \rangle\rangle = \langle \varphi \otimes \psi, R_{21}R_{12} \rangle)$, and we will write $\langle\langle \varphi, \psi \rangle\rangle = \langle \mathrm{J}(\psi), \varphi \rangle$, for some map $\mathrm{J} : \mathcal{A} \to \mathrm{U}$.

The following assertion is (in spirit) contained in [13]:

Proposition 1 J *is a morphism of right* U*-modules.*

Proof Let $\psi \in \mathcal{A}$ and $y \in \mathrm{U}$. We have:

$$\begin{aligned}
\mathrm{J}(\psi \cdot y) &= \mathrm{J}(\langle \psi, y_{(1)}\text{—}\mathrm{S}(y_{(2)})\rangle) \\
&= \langle \mathrm{id} \otimes \psi, (1 \otimes y_{(1)})R_{21}R_{12}(1 \otimes \mathrm{S}(y_{(2)}))\rangle \\
&= \langle \mathrm{id} \otimes \psi, (\mathrm{S}(y_{(1)}) \otimes 1)(y_{(2)} \otimes y_{(3)})R_{21}R_{12}(1 \otimes \mathrm{S}(y_{(4)}))\rangle \\
&= \langle \mathrm{id} \otimes \psi, (\mathrm{S}(y_{(1)}) \otimes 1)R_{21}R_{12}(y_{(2)} \otimes y_{(3)})(1 \otimes \mathrm{S}(y_{(4)}))\rangle \\
&= \langle \mathrm{id} \otimes \psi, (\mathrm{S}(y_{(1)}) \otimes 1)R_{21}R_{12}(y_{(2)} \otimes 1)\rangle \\
&= \mathrm{S}(y_{(1)})\, \mathrm{J}(\psi)\, y_{(2)} \\
&= \mathrm{J}(\psi) \cdot y.
\end{aligned}$$

□

Let Z(U) be the center of U: this is the set of invariant elements of the (adjoint) U-module U. Images under J of invariant elements of the U-module $\mathcal{A}$ thus give elements of Z(U). One can do better. A trace on U is a linear form $t \in \mathcal{A}$ such that $\forall x, y \in \mathrm{U}, t(xy) = t(yx)$; this can be written $\Delta(t) = t_{(1)} \otimes t_{(2)} = t_{(2)} \otimes t_{(1)}$.

Lemma *Let $t \in \mathcal{A}$ be a trace on* U*. Then $\langle uv^{-1}, t_{(1)}\rangle t_{(2)}$ is an invariant element of the* U*-module $\mathcal{A}$, and $\langle uv^{-1}, t_{(1)}\rangle \mathrm{J}(t_{(2)})$ is an element of* U *which commutes with any element of $\mathcal{A}^* \supseteq \mathrm{U}$.*

Proof Consider first the element $\langle uv^{-1}, t_{(1)}\rangle t_{(2)}$ of $\mathcal{A}$. The action of $x \in \mathrm{U}$ on this element is given by:

$$\begin{aligned}
t(uv^{-1}\text{—}) \cdot x &= t(uv^{-1}x_{(1)}\text{—}\mathrm{S}(x_{(2)})) \\
&= t(\mathrm{S}(x_{(2)})uv^{-1}x_{(1)}\text{—}) \\
&= t(\mathrm{S}(x_{(2)})\mathrm{S}^2(x_{(1)})uv^{-1}\text{—}) \\
&= \varepsilon(x)t(uv^{-1}\text{—}).
\end{aligned}$$

So it is an invariant element of the U-module $\mathcal{A}$.

Let now $\varphi \in \mathcal{A}^*$ and $f \in \mathcal{A}$. One has:

$$\begin{aligned}
\langle uv^{-1}, t_{(1)}\rangle\langle \mathrm{J}(t_{(2)})\varphi, f\rangle &= \langle uv^{-1}, t_{(1)}\rangle\langle f_{(1)} \otimes t_{(2)}, R_{21}R_{12}\rangle\langle \varphi, f_{(2)}\rangle \\
&= \langle uv^{-1}, t_{(1)}\rangle\langle f_{(1)} \otimes t_{(2)}, R_{21}\rangle\langle f_{(2)} \otimes t_{(3)}, R_{12}\rangle\langle \varphi, f_{(3)}t_{(4)}\mathrm{S}(t_{(5)})\rangle \\
&= \langle uv^{-1}, t_{(1)}\rangle\langle f_{(1)} \otimes t_{(2)}, R_{21}\rangle\langle f_{(3)} \otimes t_{(4)}, R_{12}\rangle\langle \varphi, t_{(3)}f_{(2)}\mathrm{S}(t_{(5)})\rangle \\
&= \langle uv^{-1}, t_{(1)}\rangle\langle f_{(2)} \otimes t_{(3)}, R_{21}\rangle\langle f_{(3)} \otimes t_{(4)}, R_{12}\rangle\langle \varphi, f_{(1)}t_{(2)}\mathrm{S}(t_{(5)})\rangle \\
&= \langle uv^{-1}, t_{(1)}\rangle\langle f_{(2)} \otimes t_{(3)}, R_{21}R_{12}\rangle\langle \varphi, f_{(1)}t_{(2)}\mathrm{S}(t_{(4)})\rangle \\
&= \langle uv^{-1}\varphi(f_{(1)}\text{—}\mathrm{S}(t_{(3)})), t_{(1)}\rangle\langle f_{(2)} \otimes t_{(2)}, R_{21}R_{12}\rangle \\
&= \langle \varphi(f_{(1)}\mathrm{S}^2(\text{—})\mathrm{S}(t_{(3)}))uv^{-1}, t_{(1)}\rangle\langle f_{(2)} \otimes t_{(2)}, R_{21}R_{12}\rangle \\
&= \langle \varphi, f_{(1)}\mathrm{S}^2(t_{(1)})\mathrm{S}(t_{(4)})\rangle\langle uv^{-1}, t_{(2)}\rangle\langle f_{(2)} \otimes t_{(3)}, R_{21}R_{12}\rangle \\
&= \langle \varphi, f_{(1)}\mathrm{S}^2(t_{(2)})\mathrm{S}(t_{(1)})\rangle\langle uv^{-1}, t_{(3)}\rangle\langle f_{(2)} \otimes t_{(4)}, R_{21}R_{12}\rangle \\
&= \langle \varphi, f_{(1)}\rangle\langle uv^{-1}, t_{(1)}\rangle\langle f_{(2)} \otimes t_{(2)}, R_{21}R_{12}\rangle \\
&= \langle uv^{-1}, t_{(1)}\rangle\langle \varphi \mathrm{J}(t_{(2)}), f\rangle.
\end{aligned}$$

Thus $\langle uv^{-1}, t_{(1)}\rangle \mathrm{J}(t_{(2)})\varphi = \langle uv^{-1}, t_{(1)}\rangle \varphi \mathrm{J}(t_{(2)})$ in $\mathcal{A}^*$ (for all $\varphi \in \mathcal{A}^*$). □

An example of such a trace t is given by the trace $\mathrm{Tr}_\mathrm{M} : (\mathrm{U} \to \mathbb{K}, x \mapsto \mathrm{Tr}(x_\mathrm{M}))$ in a f.d. U-module M. The corresponding element $\langle uv^{-1}, (\mathrm{Tr}_\mathrm{M})_{(1)}\rangle (\mathrm{Tr}_\mathrm{M})_{(2)}$ is named quantum trace and will be denoted by $\mathrm{Tr}_{q,\mathrm{M}}$. We put $z_\mathrm{M} = \mathrm{J}(\mathrm{Tr}_{q,\mathrm{M}})$. Let $\mathcal{R}(\mathrm{U})$ be the representation ring of U. If M is a f.d. U-module, Tr_M and z_M only depend of the class [M] of M in $\mathcal{R}(\mathrm{U})$.

Proposition 2 [4]: *The map* $(\mathcal{R}(\mathrm{U}) \to \mathrm{Z}(\mathrm{U}), [\mathrm{M}] \mapsto z_\mathrm{M})$ *is a well-defined ring morphism.*

The map $(\mathcal{R}(\mathrm{U}) \to \mathcal{R}(\mathrm{U}), [\mathrm{M}] \mapsto [\mathrm{M}^*])$ is well-defined and involutive, because the element u gives an U-isomorphism between a module M and its bidual M^{**} (cf. [4, §2, remark 1]). The map $(\mathrm{Z}(\mathrm{U}) \to \mathrm{Z}(\mathrm{U}), z \mapsto \mathrm{S}(z))$ is an involutive algebra morphism (at least when we restrict it to the centralizer of $\mathcal{A}^*$ in U), because S^2 is the inner automorphism of $\mathcal{A}^*$ given by u. We will show that $z_{\mathrm{M}^*} = \mathrm{S}(z_\mathrm{M})$.

We also recall that the number $\mathrm{Tr}_\mathrm{M}(uv^{-1}) = \mathrm{Tr}_{q,\mathrm{M}}(1)$ is called the quantum dimension of M and is denoted by $\dim_q \mathrm{M}$. We finally define the pairing $(\mathcal{R}(\mathrm{U}) \times \mathcal{R}(\mathrm{U}) \to \mathbb{K}, ([\mathrm{M}], [\mathrm{N}]) \mapsto \langle [\mathrm{M}], [\mathrm{N}]\rangle_1 = \langle\langle \mathrm{Tr}_{q,\mathrm{M}}, \mathrm{Tr}_{q,\mathrm{N}}\rangle\rangle)$. (This pairing gives the S-matrix considered for instance in [14, §3.1].)

Proposition 3 *1. For all* U*-module* M*,* $\dim_q \mathrm{M} = \dim_q \mathrm{M}^*$.

2. The pairing $\langle -, -\rangle_1$ *on* $\mathcal{R}(\mathrm{U})$ *is symmetric.*

3. For all $\mathrm{M} \in \mathcal{R}(\mathrm{U})$*,* $\mathrm{S}(z_\mathrm{M}) = z_{\mathrm{M}^*}$.

Proof 1. This statement appears in [14, §5.2], but the proof given there is slightly incorrect. The figure 1 presents a pictorial proof of this result. We give also an algebraic proof:

$$\begin{aligned}
\dim_q \mathrm{M} &= \langle \mathrm{Tr}_\mathrm{M}, uv^{-1}\rangle \\
&= \sum \langle \mathrm{Tr}_\mathrm{M}, \mathrm{S}(\beta_k)\alpha_k v^{-1}\rangle \\
&= \sum \langle \mathrm{Tr}_\mathrm{M}, \alpha_k \mathrm{S}(\beta_k) v^{-1}\rangle \\
&= \sum \langle \mathrm{Tr}_\mathrm{M}, \mathrm{S}(\mathrm{S}(\beta_k)\alpha_k v^{-1})\rangle \\
&= \langle \mathrm{Tr}_\mathrm{M}, \mathrm{S}(uv^{-1})\rangle = \langle \mathrm{Tr}_{\mathrm{M}^*}, uv^{-1}\rangle = \dim_q \mathrm{M}^*.
\end{aligned}$$

We have used the fact that $(\mathrm{S} \otimes \mathrm{S})(R_{12}) = R_{12}$ and the cyclicity of the trace.

2. Let M and N be two f.d. U-modules, and let Tr_M, Tr_N be the traces. We have:

$$\begin{aligned}
\langle [\mathrm{M}], [\mathrm{N}]\rangle_1 &= \langle \mathrm{Tr}_\mathrm{M} \otimes \mathrm{Tr}_\mathrm{N}, (uv^{-1} \otimes uv^{-1})(R_{21}R_{12})\rangle \\
&= \sum \langle \mathrm{Tr}_\mathrm{M} \otimes \mathrm{Tr}_\mathrm{N}, uv^{-1}\beta_i\alpha_j \otimes uv^{-1}\alpha_i\beta_j\rangle \\
&= \sum \langle \mathrm{Tr}_\mathrm{M} \otimes \mathrm{Tr}_\mathrm{N}, \mathrm{S}^2(\beta_i)uv^{-1}\alpha_j \otimes S^2(\alpha_i)uv^{-1}\beta_j\rangle \\
&= \sum \langle \mathrm{Tr}_\mathrm{M} \otimes \mathrm{Tr}_\mathrm{N}, \beta_i uv^{-1}\alpha_j \otimes \alpha_i uv^{-1}\beta_j\rangle \\
&= \sum \langle \mathrm{Tr}_\mathrm{M} \otimes \mathrm{Tr}_\mathrm{N}, uv^{-1}\alpha_j\beta_i \otimes uv^{-1}\beta_j\alpha_i\rangle \\
&= \langle \mathrm{Tr}_\mathrm{M} \otimes \mathrm{Tr}_\mathrm{N}, (uv^{-1} \otimes uv^{-1})(R_{12}R_{21})\rangle \\
&= \langle [\mathrm{N}], [\mathrm{M}]\rangle_1.
\end{aligned}$$

3. Let M be a f.d. U-modules. We compute:

$$\begin{aligned}
z_{M^*} &= \langle \mathrm{id}\otimes \mathrm{Tr}_{M^*}, (1\otimes uv^{-1})(R_{21}R_{12})\rangle \\
&= \sum \langle \mathrm{Tr}_M \circ S, \alpha_i\beta_j S(\beta_k)\alpha_k v^{-1}\rangle\ \beta_i\alpha_j \\
&= \sum \langle \mathrm{Tr}_M, \alpha_k S(\beta_k)\beta_j\alpha_i v^{-1}\rangle\ S^{-1}(\beta_i)\, S^{-1}(\alpha_j) \\
&= \sum \langle \mathrm{Tr}_M, \alpha_i\alpha_k S(\beta_k)\beta_j v^{-1}\rangle\ S^{-1}(\alpha_j\beta_i) \\
&= \sum \langle \mathrm{Tr}_M, \alpha_k\alpha_i\beta_j S(\beta_k) v^{-1}\rangle\ S^{-1}(\beta_i\alpha_j) \\
&= \sum \langle \mathrm{Tr}_M, \alpha_i\beta_j S(\beta_k)\alpha_k v^{-1}\rangle\ S^{-1}(\beta_i\alpha_j) \\
&= S^{-1}(\langle \mathrm{id}\otimes \mathrm{Tr}_M, (R_{21}R_{12})(1\otimes uv^{-1})\rangle) \\
&= S^{-1}(z_M) \;=\; S(z_M).
\end{aligned}$$

Here, we have used the Yang–Baxter identity, in the form: $(S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})(R_{13}R_{23}R_{12}^{-1}) = (S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})(R_{12}^{-1}R_{23}R_{13})$, writing: $\sum \alpha_i\alpha_k\otimes\alpha_j\beta_i\otimes S(\beta_k)\beta_j = \sum \alpha_k\alpha_i\otimes\beta_i\alpha_j\otimes\beta_j S(\beta_k)$.
□

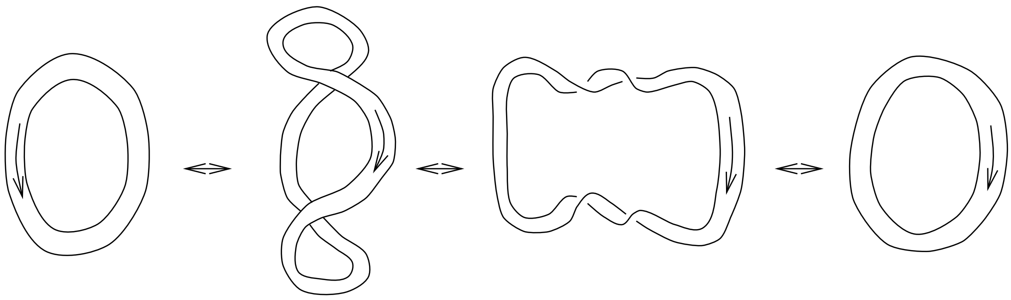

Figure 1

1.3 Reshetikhin's construction of central elements

We can reinterpret Reshetikhin's scheme as a generalization of the preceding construction. We still consider a ribbon Hopf algebra $(U, \mathcal{A}, R_{12}, v)$.

We first recall that, given a coalgebra C and an algebra D, the vector space $\mathrm{Hom}_{\mathbb{K}}(C, D)$ is an algebra for the convolution product: if $f, g \in \mathrm{Hom}_{\mathbb{K}}(C, D)$, one puts $f * g : (C \to D, x \mapsto f(x_{(1)})g(x_{(2)}))$. In our case, since the coproduct $\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ and the product $U\otimes U \to U$ are morphisms of U-modules, we can see that the space of morphisms of U-modules is a subring of the convolution algebra $\mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, U)$.

We consider the maps $J^{*p} = J * \cdots * J$ (p times), for any non-negative integer p. It is easy to verify that J^{*p} is given by $(\mathcal{A} \to U, \psi \mapsto \langle \mathrm{id}\otimes\psi, (R_{21}R_{12})^p\rangle)$. The map $(\mathcal{A} \to U, \psi \mapsto \langle \mathrm{id}\otimes\psi, (R_{21}R_{12})^{-1}\rangle)$ is an inverse for J in $\mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, U)$, therefore we may consider the maps J^{*p} for $p \in \mathbb{Z}$. These are morphisms of U-modules. If M is a f.d. U-module, we can thus define the central elements $z_M^{(p)} = J^{*p}(\mathrm{Tr}_{q,M})$; the $z_M^{(1)}$ are just our previous z_M.

Proposition 4 *1.* $z_M^{(0)} = \dim_q M$.

2. $z_M^{(-1)} = S(z_M^{(1)})$.

3. For all $\mathrm{M} \in \mathcal{R}(\mathrm{U})$, $\mathrm{S}(z_{\mathrm{M}}^{(p)}) = z_{\mathrm{M}^*}^{(p)}$.

Proof The proofs are analogous to those of the proposition 3. Let us show for instance the third assertion. Let $\mathrm{m}_{23} : \mathrm{U} \otimes \mathrm{U} \otimes \mathrm{U} \to \mathrm{U} \otimes \mathrm{U}$ be the multiplication of the two last factors. Using the Yang–Baxter equations: $R_{21}R_{23}R_{13} = R_{13}R_{23}R_{21}$ and $R_{23}R_{21}R_{31} = R_{31}R_{21}R_{23}$, we compute first:

$$\begin{aligned}
(R_{21}R_{12})^p(1 \otimes uv^{-1}) &= \sum R_{21}(1 \otimes \alpha_a \mathrm{S}(u)\beta_a)R_{12}(R_{21}R_{12})^{p-1}(1 \otimes uv^{-1}) \\
&= \mathrm{m}_{23}\big(R_{21}R_{23}(\mathrm{S}(u))_2 R_{13}(R_{31}R_{13})^{p-1}(uv^{-1})_3\big) \\
&= \mathrm{m}_{23}\big(R_{13}(R_{31}R_{13})^{p-1}R_{23}R_{21}(\mathrm{S}(u))_2(uv^{-1})_3\big) \\
&= \sum (1 \otimes \alpha_a\alpha_b \mathrm{S}(u))R_{12}(R_{21}R_{12})^{p-1}(\beta_b \otimes \beta_a uv^{-1}).
\end{aligned}$$

Then for any f.d. U-module M:

$$\begin{aligned}
\mathrm{S}(z_{\mathrm{M}}^{(p)}) = \mathrm{S}^{-1}(z_{\mathrm{M}}^{(p)}) &= \langle \mathrm{S}^{-1} \otimes \mathrm{Tr}_{\mathrm{M}}, (R_{21}R_{12})^p(1 \otimes uv^{-1})\rangle \\
&= \sum \langle \mathrm{S}^{-1} \otimes \mathrm{Tr}_{\mathrm{M}}, (1 \otimes \alpha_a\alpha_b\mathrm{S}(u))R_{12}(R_{21}R_{12})^{p-1}(\beta_b \otimes \beta_a uv^{-1})\rangle \\
&= \sum \langle \mathrm{S}^{-1} \otimes \mathrm{Tr}_{\mathrm{M}}, R_{12}(R_{21}R_{12})^{p-1}(\beta_b \otimes \beta_a u \alpha_a\alpha_b \mathrm{S}(u)v^{-1})\rangle \\
&= \sum \langle \mathrm{S}^{-1} \otimes \mathrm{Tr}_{\mathrm{M}}, R_{12}(R_{21}R_{12})^{p-1}(\beta_b \otimes \alpha_b \mathrm{S}(u)v^{-1})\rangle \\
&= \langle \mathrm{S}^{-1} \otimes \mathrm{Tr}_{\mathrm{M}}, (R_{12}R_{21})^p(1 \otimes \mathrm{S}(u)v^{-1})\rangle \\
&= \langle \mathrm{id} \otimes \mathrm{Tr}_{\mathrm{M}^*}, (\mathrm{S}^{-1} \otimes \mathrm{S}^{-1})((R_{12}R_{21})^p(1 \otimes \mathrm{S}(u)v^{-1}))\rangle \\
&= \langle \mathrm{id} \otimes \mathrm{Tr}_{\mathrm{M}^*}, (1 \otimes uv^{-1})(R_{21}R_{12})^p\rangle \\
&= z_{\mathrm{M}^*}^{(p)}\ .
\end{aligned}$$

□

We will study in more details these elements $z_{\mathrm{M}}^{(p)}$ in the case of a quantized enveloping algebra. Let us just point out that Reshetikhin discovered them in a graphical way. In his language, $z_{\mathrm{M}}^{(2)}$ is the tangle invariant corresponding to the colored (ribbon) tangle pictured in figure 2, and $\mathrm{Tr}_{q,\mathrm{N}}(z_{\mathrm{M}}^{(-1)})$ is (for any f.d. U-module N) the invariant of the (ribbon) tangle shown in the figure 3.

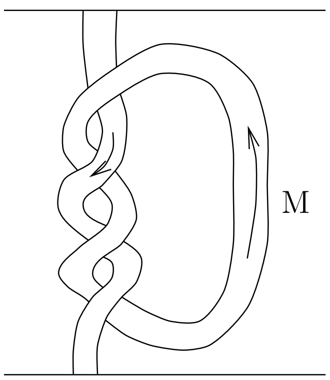

Figure 2

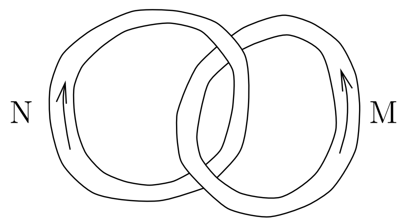

Figure 3

2 The center of $\mathrm{U}_q\mathfrak{g}$

2.1 Joseph and Letzter's construction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $\mathrm{Q} \subseteq \mathrm{P} \subseteq \mathfrak{h}^*$ the root and the weight lattices, W the Weyl group, $(\cdot|\cdot) : \mathrm{P} \times \mathrm{P} \to \mathbb{Z}$ a $\mathbb{Z}$-valued W-invariant non-degenerate bilinear form. We choose a set $\{\alpha_1, \dots, \alpha_\ell\}$ of simple roots and put $d_i = \frac{\|\alpha_i\|^2}{2}$. Let P_{++} be the set of dominant weights, R_+ be the set of positive roots, $\rho \in \mathrm{P}_{++}$ be half the sum of the positive roots, w_0 be the longest element of W, and $\varepsilon : \mathrm{W} \to \{\pm 1\}$ be the signature function.

Let q be an indeterminate. $\mathrm{U}_q\mathfrak{g}$ is defined to be the $\mathbb{C}(q)$-algebra generated by elements K_λ ($\lambda \in \mathrm{P}$), E_i and F_i ($i \in \{1, \dots, \ell\}$) with some well-known relations, among which: $K_\lambda E_i = q^{(\lambda|\alpha_i)} E_i K_\lambda$, $K_\lambda F_i = q^{-(\lambda|\alpha_i)} F_i K_\lambda$, $E_i F_j - F_j E_i = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}}$. $\mathrm{U}_q\mathfrak{g}$ is a Hopf algebra, the coproduct being given by $\Delta(K_\lambda) = K_\lambda \otimes K_\lambda$, $\Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i$, $\Delta(F_i) = F_i \otimes K_{-\alpha_i} + 1 \otimes F_i$. Let $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$ be the center of $\mathrm{U}_q\mathfrak{g}$.

Joseph and Letzter have studied the adjoint module $\mathrm{U}_q\mathfrak{g}$. We state a consequence of their results for the right adjoint action introduced in section 1.2. The set of all ad-finite elements in $\mathrm{U}_q\mathfrak{g}$ is a subalgebra $\mathrm{F}(\mathrm{U}_q\mathfrak{g})$ of $\mathrm{U}_q\mathfrak{g}$. There is a natural decomposition in blocks: $\mathrm{F}(\mathrm{U}_q\mathfrak{g}) = \bigoplus_{\lambda \in \mathrm{P}_{++}} (K_{2\lambda}) \cdot \mathrm{U}_q\mathfrak{g}$, and each block $(K_{2\lambda}) \cdot \mathrm{U}_q\mathfrak{g}$ contains a unique line defining the trivial $\mathrm{U}_q\mathfrak{g}$-module (see [8, § 4.13]). We denote by $\mathcal{Z}_{\lambda+\rho}$ the corresponding central element of $\mathrm{U}_q\mathfrak{g}$. Hence the center of $\mathrm{U}_q\mathfrak{g}$ has the family $(\mathcal{Z}_{\lambda+\rho})_{\lambda \in \mathrm{P}_{++}}$ for $\mathbb{C}(q)$-basis.

2.2 The Harish-Chandra map

Let U^+, U^0 and U^- be the subalgebras of $\mathrm{U}_q\mathfrak{g}$ generated by the $(E_i)_{1 \le i \le \ell}$, $(K_\lambda)_{\lambda \in \mathrm{P}}$ and $(F_i)_{1 \le i \le \ell}$ respectively. The multiplication induces an isomorphism of vector spaces $\mathrm{U}_q\mathfrak{g} \simeq \mathrm{U}^+ \otimes \mathrm{U}^0 \otimes \mathrm{U}^-$. Let $\Psi : (\mathrm{U}_q\mathfrak{g} \to \mathrm{U}^0)$ be the Harish-Chandra map given by $(E\, K_\lambda\, F \mapsto q^{(\lambda|\rho)} \varepsilon(E) \varepsilon(F) K_{w_0\lambda})$ in this triangular decomposition of $\mathrm{U}_q\mathfrak{g}$. There is a natural isomorphism from the algebra of the weight group onto a subalgebra of U^0, given by $\tau : (\mathbb{C}(q)[\mathrm{P}] \to \mathrm{U}^0, e^\lambda \mapsto K_{2\lambda})$; there is thus a natural action of W on $(\operatorname{im} \tau)$. Rosso [15] (see also [16, §2.8]) has shown that Ψ defines an isomorphism from the center of $\mathrm{U}_q\mathfrak{g}$ to the set of W-invariant elements in $(\operatorname{im} \tau)$, and thus that $\tau^{-1} \circ \Psi$ defines an isomorphism from $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$ to the set of exponential invariants $(\mathbb{C}(q)[\mathrm{P}])^{\mathrm{W}}$.

We will consider only $\mathrm{U}_q\mathfrak{g}$-modules which are f.d. and of type 1 (following the terminology of Chari and Pressley [2, p. 314]), that is, modules M such that $\mathrm{M} = \bigoplus_{\lambda \in \mathrm{P}} \{x \in \mathrm{M} \mid \forall \mu \in \mathrm{P},\ K_\mu \cdot x = q^{(\lambda|\mu)} x\}$. The f.d. modules of type 1 are completely reducible and the simple ones are classified by their highest weight $\lambda \in \mathrm{P}_{++}$. These will be denoted by $\mathrm{L}(\lambda)$. The dual of $\mathrm{L}(\lambda)$ will be identified with $\mathrm{L}(\lambda^*)$, where $\lambda^* = -w_0(\lambda)$. The Grothendieck ring $\mathcal{R}$ of the category of f.d. type 1 modules is naturally isomorphic to the representation ring of $\mathfrak{g}$, and the formal character ch gives an isomorphism from $\mathcal{R}$ to the algebra of exponential invariants $\mathbb{Z}[\mathrm{P}]^{\mathrm{W}}$. Finally each module $\mathrm{L}(\lambda)$ is absolutely simple, so has a central character $\chi_{\lambda+\rho}$. If ev_λ denotes the algebra morphism $(\mathrm{U}^0 \to \mathbb{C}(q), K_\mu \mapsto q^{(\lambda|\mu)})$, one has $\chi_\lambda(z) = \mathrm{ev}_\lambda \circ \Psi(z)$ for all $z \in \mathrm{Z}(\mathrm{U}_q\mathfrak{g})$.

2.3 Expression of the $z^{(1)}_{\mathrm{M}}$

The most important thing for us is that $\mathrm{U}_q\mathfrak{g}$ is a quasitriangular Hopf algebra [3]. There exist two possible R-matrices [7]: we choose the one with the structure $R_{12} = \sum$(diagonal part)(polynomial in F) $\otimes$ (polynomial in E), and the other one is R_{21}^{-1}. The element u is such that $u\,\mathrm{S}(u)^{-1} = K_{-4\rho}$; there is a natural square root of this group-like element, and we define the ribbon element v by the equation $uv^{-1} = K_{-2\rho}$. Finally, R_{12} acts only in f.d. type 1 $\mathrm{U}_q\mathfrak{g}$-modules, and we define consequently $\mathcal{A}_q\mathrm{G}$ as the $\mathbb{C}(q)$-linear span of the matrix coefficients of these representations. Then $\mathcal{A}_q\mathrm{G}$ separates the points of $\mathrm{U}_q\mathfrak{g}$, and one gets a ribbon Hopf algebra $(\mathrm{U}_q\mathfrak{g}, \mathcal{A}_q\mathrm{G}, R_{12}, v)$. The constructions of section 1.3 allow us to consider the central elements $z^{(p)}_{\mathrm{M}}$ for any type 1 f.d. $\mathrm{U}_q\mathfrak{g}$-module M.

Proposition 5 [8, § 6.10][5]: *Up to a scalar, one has $\mathcal{Z}_{\lambda+\rho} = z^{(1)}_{\mathrm{L}(\lambda)}$. And one has $\Psi(z^{(1)}_{\mathrm{L}(\lambda)}) = \tau(\mathrm{ch}\,\mathrm{L}(\lambda))$.*

Proof Note first that for any $\mu \in \mathrm{P}_{++}$, one has $\dim_q \mathrm{L}(\mu) = \dim_q \mathrm{L}(\mu^*) = \mathrm{Tr}_{\mathrm{L}(\mu)}(K_{2\rho}) = \mathrm{Tr}_{\mathrm{L}(\mu)}(K_{-2\rho}) = \prod_{\alpha\in\mathrm{R}_+} \frac{q^{(\alpha|\mu+\rho)} - q^{-(\alpha|\mu+\rho)}}{q^{(\alpha|\rho)} - q^{-(\alpha|\rho)}}$ is different from 0.

Let $\lambda \in \mathrm{P}_{++}$; we can write $\mathcal{Z}_{\lambda+\rho} = \mathrm{S}(x_{(1)})K_{2\lambda}x_{(2)}$ for some $x \in \mathrm{U}_q\mathfrak{g}$, by the definition of $\mathcal{Z}_{\lambda+\rho}$. We then compute, for any $\mu \in \mathrm{P}_{++}$:

$$\begin{aligned}\dim_q \mathrm{L}(\mu)\,\chi_{\mu+\rho}(\mathcal{Z}_{\lambda+\rho}) &= \mathrm{Tr}_{\mathrm{L}(\mu)}(K_{2\rho}\mathcal{Z}_{\lambda+\rho})\\ &= \mathrm{Tr}_{\mathrm{L}(\mu)}(K_{2\rho}\mathrm{S}(x_{(1)})K_{2\lambda}x_{(2)})\\ &= \mathrm{Tr}_{\mathrm{L}(\mu)}(x_{(2)}\mathrm{S}^{-1}(x_{(1)})K_{2\rho}K_{2\lambda})\\ &= \varepsilon(x)\,\mathrm{Tr}_{\mathrm{L}(\mu)}(K_{2(\lambda+\rho)}).\end{aligned}$$

Using the weight decomposition of $\mathrm{L}(\mu)$ and the Weyl character formula, we find that:

$$\begin{aligned}\dim_q \mathrm{L}(\mu)\,\chi_{\mu+\rho}(\mathcal{Z}_{\lambda+\rho}) &= \varepsilon(x)\,\frac{\sum_{w\in\mathrm{W}}\varepsilon(w)q^{2(\lambda+\rho|w(\mu+\rho))}}{\sum_{w\in\mathrm{W}}\varepsilon(w)q^{2(\lambda+\rho|w\rho)}}\\ &= \frac{\varepsilon(x)}{\dim_q \mathrm{L}(\lambda)}\,\frac{\sum_{w\in\mathrm{W}}\varepsilon(w)q^{2(\lambda+\rho|w(\mu+\rho))}}{\sum_{w\in\mathrm{W}}\varepsilon(w)q^{2(\rho|w\rho)}}.\end{aligned}$$

According to [18, lemma 3.5.1], we thus have:

$$\dim_q \mathrm{L}(\mu)\,\chi_{\mu+\rho}(\mathcal{Z}_{\lambda+\rho}) = \frac{\varepsilon(x)}{\dim_q \mathrm{L}(\lambda)}\,\langle\!\langle \mathrm{Tr}_{q,\mathrm{L}(\mu)}, \mathrm{Tr}_{q,\mathrm{L}(\lambda)}\rangle\!\rangle = \frac{\varepsilon(x)}{\dim_q \mathrm{L}(\lambda)}\,\langle \mathrm{Tr}_{q,\mathrm{L}(\mu)}, z^{(1)}_{\mathrm{L}(\lambda)}\rangle.$$

$\mathcal{Z}_{\lambda+\rho}$ and $\frac{\varepsilon(x)}{\dim_q \mathrm{L}(\lambda)} z^{(1)}_{\mathrm{L}(\lambda)}$ act therefore by the same scalar in any simple f.d. type 1 $\mathrm{U}_q\mathfrak{g}$-module, thus are equal.
On the other hand:

$$\begin{aligned}\chi_{\mu+\rho}(z^{(1)}_{\mathrm{L}(\lambda)}) &= \frac{1}{\dim_q \mathrm{L}(\mu)}\,\langle \mathrm{Tr}_{q,\mathrm{L}(\mu)}, z^{(1)}_{\mathrm{L}(\lambda)}\rangle\\ &= \frac{1}{\dim_q \mathrm{L}(\mu)}\,\frac{\sum_{w\in\mathrm{W}}\varepsilon(w)q^{2(\lambda+\rho|w(\mu+\rho))}}{\sum_{w\in\mathrm{W}}\varepsilon(w)q^{2(\rho|w\rho)}} \qquad (2)\\ &= \frac{\sum_{w\in\mathrm{W}}\varepsilon(w)q^{2(\lambda+\rho|w(\mu+\rho))}}{\sum_{w\in\mathrm{W}}\varepsilon(w)q^{2(\mu+\rho|w\rho)}} = \mathrm{ev}_{\mu+\rho}(\tau(\mathrm{ch}\,\mathrm{L}(\lambda))),\end{aligned}$$

so that $\Psi(z^{(1)}_{\mathrm{L}(\lambda)}) = \tau(\mathrm{ch}\,\mathrm{L}(\lambda))$. □

2.4 Expression of the $z^{(p)}_{\mathrm{M}}$

Now, we can prove the main result of this article. Because of the preceding description, it is natural to put $\mathcal{Z}^{(p)}_{\lambda+\rho} = z^{(p)}_{\mathrm{L}(\lambda)}$, where the $z^{(p)}_{\mathrm{M}}$ were defined (in section 1.3) as $\langle \mathrm{id} \otimes \mathrm{Tr}_{q,\mathrm{M}}, (R_{21}R_{12})^p\rangle$. This defines $\mathcal{Z}^{(p)}_{\lambda+\rho}$ if $\lambda+\rho$ is a regular dominant weight; we define $\mathcal{Z}^{(p)}_{\mu}$ for any weight μ by the requirement $\mathcal{Z}^{(p)}_{w\mu} = \varepsilon(w)\mathcal{Z}^{(p)}_{\mu}$ for any $w \in \mathrm{W}$; in particular, $\mathcal{Z}^{(p)}_{\mu} = 0$ if μ belongs to a wall, and $\mathcal{Z}^{(p)}_{0} = 0$, $\mathcal{Z}^{(p)}_{\rho} = 1$.

Theorem 1 *For any* $\lambda \in \mathrm{P}$, $\sum_{w\in\mathrm{W}} \varepsilon(w) q^{2p(\lambda|w\rho)} \mathcal{Z}^{(p)}_{\lambda+w\rho} = \sum_{w\in\mathrm{W}} \varepsilon(w) q^{2(\lambda|w\rho)} \mathcal{Z}^{(1)}_{p\lambda+w\rho}$. *These relations caracterize the elements* $\mathcal{Z}^{(p)}_{\lambda}$.

Proof We will check that any central character $\chi_{\mu+\rho}$ takes the same value on both sides of this equation. For $\lambda \in \rho + \mathrm{P}_{++}$ and $w \in \mathrm{W}$, we set (in the Grothendieck ring $\mathcal{R}$): $\mathrm{L}(w\lambda-\rho) = \varepsilon(w)\mathrm{L}(\lambda-\rho)$; it is just a convenient convention allowing to write the formula for the tensor product multiplicities in $\mathcal{R}$:

$$\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu) = \textstyle\sum_{\nu\in\mathrm{P}_{++}} \left(\sum_{w\in\mathrm{W}} \varepsilon(w) \dim \mathrm{L}(\lambda)_{w(\nu+\rho)-\mu-\rho}\right) \mathrm{L}(\nu)$$

(where $\mathrm{L}(\lambda)_\tau$ is the subspace of $\mathrm{L}(\lambda)$ of weight τ) in the following simple form:

$$\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu) = \textstyle\sum_{\nu\in\mathrm{P}} (\dim \mathrm{L}(\lambda)_{\nu-\mu})\, \mathrm{L}(\nu) = \sum_{\nu\in\mathrm{P}} (\dim \mathrm{L}(\lambda)_{\nu})\, \mathrm{L}(\mu+\nu).$$

We similarly set $\mathrm{Tr}_{q,\mathrm{L}(w\lambda-\rho)} = \varepsilon(w)\,\mathrm{Tr}_{q,\mathrm{L}(\lambda-\rho)}$, so that $(\mathcal{R} \to \mathcal{A}_q\mathrm{G}, \mathrm{L}(\lambda) \mapsto \mathrm{Tr}_{q,\mathrm{L}(\lambda)})$ is a well-defined ring homomorphism. We also need the ribbon element $v = uK_{2\rho}$. It acts in the module $\mathrm{L}(\lambda)$ as the scalar $q^{-(\lambda|\lambda+2\rho)}$. The formula (1) (in the section 1.1) shows that $(R_{21}R_{12})$ acts as the scalar $q^{(\nu|\nu+2\rho)-(\lambda|\lambda+2\rho)-(\mu|\mu+2\rho)}$ in the isotypical component of type $\mathrm{L}(\nu)$ of the tensor product $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$.

Then we compute:

$$\begin{aligned}
\dim_q \mathrm{L}(\mu)\chi_{\mu+\rho}(\text{l.h.s.}) &= \mathrm{Tr}_{q,\mathrm{L}(\mu)}(\text{l.h.s.}) \\
&= \textstyle\sum_{w\in\mathrm{W}} \varepsilon(w) q^{2p(\lambda|w\rho)} \langle \mathrm{Tr}_{q,\mathrm{L}(\mu)} \otimes \mathrm{Tr}_{q,\mathrm{L}(\lambda+w\rho-\rho)}, (R_{21}R_{12})^p\rangle \\
&= \textstyle\sum_{w\in\mathrm{W}} \varepsilon(w) q^{2p(\lambda|w\rho)} \Big(\sum_{\nu\in\mathrm{P}} \dim \mathrm{L}(\lambda+w\rho-\rho)_\nu \\
&\qquad \langle \mathrm{Tr}_{q,\mathrm{L}(\mu+\nu)}, q^{p((\mu+\nu|\mu+\nu+2\rho)-(\lambda+w\rho-\rho|\lambda+w\rho+\rho)-(\mu|\mu+2\rho))}\rangle \Big) \\
&= \textstyle\sum_{\nu\in\mathrm{P}} \left(\sum_{w\in\mathrm{W}} \varepsilon(w) \dim \mathrm{L}(\lambda+w\rho-\rho)_\nu \right) \dim_q \mathrm{L}(\mu+\nu)\; q^{p((\nu|\nu+2\mu+2\rho)-\|\lambda\|^2)} \\
&= \textstyle\sum_{w'\in\mathrm{W}} \dim_q \mathrm{L}(w'\lambda+\mu)\; q^{2p(w'\lambda|\mu+\rho)},
\end{aligned}$$

since $\sum_{w\in\mathrm{W}} \varepsilon(w)\,\mathrm{ch}\,\mathrm{L}(\lambda+w\rho-\rho) = \sum_{w'\in\mathrm{W}} e^{w'\lambda}$, a consequence of the Weyl character formula.

Setting $D = \sum_{w\in \mathrm{W}} \varepsilon(w) q^{2(\rho|w\rho)}$, we compute, using formula (2):

$$\begin{aligned}\dim_q \mathrm{L}(\mu)\, \chi_{\mu+\rho}(\text{r.h.s.}) &= \textstyle\sum_{w\in \mathrm{W}} \varepsilon(w) q^{2(\lambda|w\rho)} \dim_q \mathrm{L}(\mu) \chi_{\mu+\rho}(\mathcal{Z}^{(1)}_{p\lambda+w\rho}) \\ &= \tfrac{1}{D} \textstyle\sum_{w\in \mathrm{W}} \varepsilon(w) q^{2(\lambda|w\rho)} \left(\sum_{w'\in \mathrm{W}} \varepsilon(w') q^{2(\mu+\rho|w'(p\lambda+w\rho))}\right) \\ &= \tfrac{1}{D} \textstyle\sum_{w,w'\in \mathrm{W}} \varepsilon(w) q^{2(\mu+\rho|w'p\lambda)} q^{2(w'\lambda+\mu+\rho|w\rho)} \\ &= \textstyle\sum_{w'\in \mathrm{W}} q^{2p(w'\lambda|\mu+\rho)} \dim_q \mathrm{L}(w'\lambda+\mu).\end{aligned}$$

Hence our formula holds. The remainder of this section will prove that it fully caracterizes the elements $\mathcal{Z}^{(p)}_\lambda$. $\square$

To conclude the proof of the theorem, we use the following proposition, formulated in an abstract setting. Let V be a $\mathbb{Q}$-vector space. The set of functions $\mathrm{P} \to \mathrm{V}$ is in bijection with $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathrm{P}], \mathrm{V})$. By the action of W on $\mathbb{Z}[\mathrm{P}]$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathrm{P}], \mathrm{V})$ becomes a W-module, and we note H_1 and H_ε the isotypical components corresponding to the characters 1 and ε of W.

Proposition 6 *If* $v \in \mathrm{Aut}_{\mathbb{Q}}(\mathrm{V})$, *then the map* $(\mathrm{H}_\varepsilon \to \mathrm{H}_1, f \mapsto (\lambda \mapsto \sum_{w\in \mathrm{W}} \varepsilon(w) v^{(\lambda|w\rho)} f(\lambda + w\rho)))$ *is injective.*

Proof Let $f \in \mathrm{H}_\varepsilon$ belonging to the kernel of the map, and suppose that f is non-zero. Since $f \in \mathrm{H}_\varepsilon$, $f(0) = 0$. Let $\lambda \in \mathrm{P}_{++}$ be a dominant weight such that $f(\lambda+\rho) \neq 0$ with $\|\lambda+\rho\|$ minimal for this property. By hypothesis, $\sum_{w\in \mathrm{W}} \varepsilon(w) v^{(\lambda|w\rho)} f(\lambda+w\rho) = 0$. We rewrite this sum $\sum_{w\in \mathrm{W}} v^{(w\lambda|\rho)} f(w\lambda+\rho) = |\mathrm{W}_\lambda| \sum_{w\in \mathrm{W}/\mathrm{W}_\lambda} v^{(w\lambda|\rho)} f(w\lambda+\rho)$ where W_λ is the stabilizer of λ in W. Thus $\sum_{w\in \mathrm{W}/\mathrm{W}_\lambda} v^{(w\lambda|\rho)} f(w\lambda+\rho) = 0$. The weights occurring in this sum verify $\|w\lambda+\rho\| < \|\lambda+\rho\|$ if $w \neq \mathrm{W}_\lambda$. (To see this, write a reduced decomposition of (the shortest representative of) w: $w = s_{i_1} \cdots s_{i_k}$, with $s_{i_k}(\lambda) \neq \lambda$. Then $\{\beta \in \mathrm{R}_+ \mid w(\beta) \in -\mathrm{R}_+\} = \{\alpha_{i_k}, s_{i_k}(\alpha_{i_{k-1}}), \dots, s_{i_k} s_{i_{k-1}} \cdots s_{i_2}(\alpha_{i_1})\}$, so $\rho - w^{-1}(\rho) \in \alpha_{i_k} + \mathrm{Q}_+$, where Q_+ is the $\mathbb{Z}_+$-span of R_+. So $(\lambda \mid \rho) > (\lambda \mid w^{-1}\rho)$, hence $\|w\lambda+\rho\| < \|\lambda+\rho\|$.) This contradicts the choice of λ. $\square$

An application of this proposition proves the last assertion in our theorem 1.

Remarks 1. Drinfel′d's result (proposition 2) implies that the $\mathcal{Z}^{(1)}_{\mu+\rho}$ $(\mu \in \mathrm{P}_{++})$ span a $\mathbb{Z}$-form in $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$. We then have that the $\mathcal{Z}^{(p)}_\lambda$ belong to $\sum_{\mu\in \mathrm{P}_{++}} \mathbb{Z}[q,q^{-1}] \mathcal{Z}^{(1)}_{\mu+\rho}$. We take now the image of our formula in $\mathbb{Z}[q,q^{-1}][\mathrm{P}]^{\mathrm{W}}$ by the map $\tau^{-1} \circ \Psi$, and we compose by the evaluation at $q = 1$: $\mathbb{Z}[q,q^{-1}][\mathrm{P}] \to \mathbb{Z}[\mathrm{P}]$. We know that $\mathcal{Z}^{(1)}_{\lambda+\rho}$ is sent to $\mathrm{ch}\,\mathrm{L}(\lambda)$, and the Weyl character formula then tells that $\mathcal{Z}^{(p)}_{\lambda+\rho}$ is sent to $\psi^p(\mathrm{ch}\,\mathrm{L}(\lambda))$ [17]. Here ψ^p is the Adams operator in the algebra $\mathbb{Z}[\mathrm{P}]$: it is the algebra morphism $e^\lambda \mapsto e^{p\lambda}$; it may be viewed as the convolution product $\mathrm{id} * \cdots * \mathrm{id}$ (p times), $\mathbb{Z}[\mathrm{P}]$ being a Hopf algebra (over $\mathbb{Z}$).

2. The scalar products on $\mathcal{R}$ given by $\langle \mathrm{L}(\lambda), \mathrm{L}(\mu)\rangle_p = \langle \mathrm{Tr}_{q,\mathrm{L}(\lambda)}, z^{(p)}_{\mathrm{L}(\mu)}\rangle$ satisfy $\langle \mathrm{L}(\lambda), \mathrm{L}(\mu)\rangle_p = \langle \mathrm{L}(\mu), \mathrm{L}(\lambda)\rangle_p = \langle \mathrm{L}(\lambda^*), \mathrm{L}(\mu^*)\rangle_p$. They are similar to Macdonald's scalar products (cf. [10]), but the combinatorics is considerably much easier in our case.

3 An application to a theorem of Faddeev, Reshetikhin and Takhtadzhyan

Besides being funny for itself, our formula (theorem 1) has two consequences. First, it gives relations between invariants of knots that one constructs with the quantum groups. It can also be applied to prove a theorem stated in [6].

Let us suppose that $\mathfrak{g}$ is of classical type A, B, C or D and of rank ℓ. We adopt Bourbaki's conventions for the root systems [1]. For instance, $\mathrm{L}(\varpi_1)$ is the natural representation of $\mathfrak{g}$. We will keep track of the normalization of the invariant bilinear form on $\mathfrak{h}^*$ by letting d be half of the square of the length of a short root.

We now recall the formalism of [6]. We choose a basis (v_i) of $\mathrm{L}(\varpi_1)$, consisting of vectors of weights (λ_i), we let (v_i^*) be the dual basis, and we consider the matrix $T = (t_{ij})$ of the coefficients of this representation: for $x \in \mathrm{U}_q\mathfrak{g}$, $\langle t_{ij}, x\rangle = \langle v_i^*, x \cdot v_j\rangle$. Faddeev, Reshetikhin and Takhtadzhyan define the matrices with coefficients in $\mathrm{U}_q\mathfrak{g}$: $L^{\pm} = (l_{ij}^{\pm})$, with $l_{ij}^+ = \langle \mathrm{id} \otimes t_{ij}, R_{12}\rangle \in \mathrm{U}^0\mathrm{U}^-$ and $\mathrm{S}(l_{ij}^-) = \langle t_{ij} \otimes \mathrm{id}, R_{12}\rangle \in \mathrm{U}^+\mathrm{U}^0$. They consider the elements of $\mathrm{U}_q\mathfrak{g}$:

$$\begin{aligned}
\mathrm{Tr}(q^{2\rho}(L^+\mathrm{S}(L^-))^k) &= \sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_k}} q^{(2\rho|\lambda_{i_1})} l^+_{i_1j_1} S(l^-_{j_1i_2}) \cdots l^+_{i_kj_k} S(l^-_{j_ki_1}) \\
&= \sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_k}} q^{(2\rho|\lambda_{i_1})} \langle \mathrm{id} \otimes t_{i_1j_1}, R_{12}\rangle \langle t_{j_1i_2} \otimes \mathrm{id}, R_{12}\rangle \cdots \\
&\qquad\qquad \langle \mathrm{id} \otimes t_{i_kj_k}, R_{12}\rangle \langle t_{j_ki_1} \otimes \mathrm{id}, R_{12}\rangle \\
&= \sum_{i_1} q^{(2\rho|\lambda_{i_1})} \langle t_{i_1i_1} \otimes \mathrm{id}, (R_{21}R_{12})^k\rangle \\
&= \langle \mathrm{Tr}_{\mathrm{L}(\varpi_1)} \otimes \mathrm{id}, (R_{21}R_{12})^k (K_{2\rho} \otimes 1)\rangle \\
&= \langle \mathrm{Tr}_{\mathrm{L}(\varpi_1)^*} \otimes \mathrm{S}, (\mathrm{S}^{-1} \otimes \mathrm{S}^{-1})((R_{21}R_{12})^k (K_{2\rho} \otimes 1))\rangle \\
&= \langle \mathrm{S} \otimes \mathrm{Tr}_{\mathrm{L}(\varpi_1)^*}, (1 \otimes K_{-2\rho})(R_{21}R_{12})^k\rangle \\
&= \mathrm{S}(J^{*k}(\mathrm{Tr}_{q,\mathrm{L}(\varpi_1)^*})) \\
&= \mathrm{S}(z^{(k)}_{\mathrm{L}(\varpi_1)^*}) \\
&= z^{(k)}_{\mathrm{L}(\varpi_1)}.
\end{aligned}$$

They state ([6, theorem 14]) that these elements belong to and generate the center of $\mathrm{U}_q\mathfrak{g}$. We will indeed prove a more precise theorem:

Theorem 2 *Let* Y *be the subalgebra of* $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$ *generated by the elements* $z^{(1)}_{\mathrm{L}(\varpi_1)}, \dots, z^{(\ell)}_{\mathrm{L}(\varpi_1)}$. *Then:*

- *in case* A_ℓ *or* C_ℓ, Y *is the whole algebra* $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$.
- *in case* B_ℓ, *we use the Harish-Chandra map* $\tau^{-1} \circ \Psi : \mathrm{Z}(\mathrm{U}_q\mathfrak{g}) \tilde{\to} (\mathbb{C}(q)[\mathrm{P}])^{\mathrm{W}}$ *to describe* $\mathrm{Z}(\mathrm{U}_q\mathfrak{g})$. Y *corresponds to the subalgebra spanned by the characters* $\mathrm{ch}\,\mathrm{M} \in \mathbb{Z}[\mathrm{P}]^{\mathrm{W}}$ *of vectorial representations of* $\mathfrak{g}$. *It is the subalgebra of* $(\mathbb{C}(q)[\mathrm{P}])^{\mathrm{W}}$ *fixed by the involution* $s : (\mathrm{ch}\,\mathrm{L}(\varpi_i) \mapsto \mathrm{ch}\,\mathrm{L}(\varpi_i)\ (i \leq \ell-1), \mathrm{ch}\,\mathrm{L}(\varpi_\ell) \mapsto -\,\mathrm{ch}\,\mathrm{L}(\varpi_\ell))$.

- *in case* D_ℓ, Y *corresponds to the subalgebra of* $(\mathbb{C}(q)[P])^W$ *fixed by the involutions* $s : (\operatorname{ch} L(\varpi_i) \mapsto \operatorname{ch} L(\varpi_i)\ (i \le \ell-2), \operatorname{ch} L(\varpi_{\ell-1}) \leftrightarrow \operatorname{ch} L(\varpi_\ell))$ *and* $t : (\operatorname{ch} L(\varpi_i) \mapsto \operatorname{ch} L(\varpi_i)\ (i \le \ell-2), \operatorname{ch} L(\varpi_{\ell-1}) \leftrightarrow -\operatorname{ch} L(\varpi_\ell))$.

Proof We rewrite our formula in the form:

$$\sum_{w\in W} \varepsilon(w)\, q^{2p(\lambda|w^{-1}\rho)}\, \mathcal{Z}^{(p)}_{\lambda+w^{-1}\rho} = \sum_{w\in W} \varepsilon(w)\, q^{2(\lambda|w^{-1}\rho)}\, \mathcal{Z}^{(1)}_{p\lambda+w^{-1}\rho},$$

in which we can replace the sums over W by sums over W/W_λ. We will only consider the cases A_ℓ and B_ℓ (the cases C_ℓ and D_ℓ being similar).

- Case A_ℓ: the shortest representatives of elements in W/W_{ϖ_1} are given by $\{1, s_1, s_2s_1, \ldots, s_\ell\cdots s_1\}$. Let $1 \le p \le \ell$. For $1 \le k \le \ell+1$ and $w = s_{k-1}\cdots s_1$, one has:

  - $\varpi_1 + w^{-1}\rho$ belongs to a wall iff $k > 1$, and it is $\varpi_1+\rho$ otherwise;
  - $p\varpi_1 + w^{-1}\rho$ belongs to a wall iff $k > p$, and it is $(p-k)\varpi_1 + \varpi_k + \rho$ if $k \le p$;
  - $(\varpi_1 \mid w^{-1}\rho) = d(\frac{\ell}{2} - k + 1)$ ($d = 1$ in Bourbaki's normalization).

  Then our formula implies that :

  $$q^{d\cdot 2p\frac{\ell}{2}}\, \mathcal{Z}^{(p)}_{\varpi_1+\rho} = \sum_{k=1}^{p} (-1)^{k+1}\, q^{d\cdot 2(\frac{\ell}{2}-k+1)}\, \mathcal{Z}^{(1)}_{(p-k)\varpi_1+\varpi_k+\rho}.$$

  The rule for tensor product multiplicities [9] and the proposition 2 show that, for $1 \le n \le \ell-1$ and $1 \le m$:

  $$\mathcal{Z}^{(1)}_{m\varpi_1+\rho} \cdot \mathcal{Z}^{(1)}_{\varpi_n+\rho} = \mathcal{Z}^{(1)}_{m\varpi_1+\varpi_n+\rho} + \mathcal{Z}^{(1)}_{(m-1)\varpi_1+\varpi_{n+1}+\rho}.$$

  Let $Y_p \subseteq Z(U_q\mathfrak{g})$ be the $\mathbb{C}(q)$ subalgebra generated by the $(\mathcal{Z}^{(s)}_{\varpi_1+\rho})_{1\le s\le p}$. By induction on p, we show that for $1 \le p \le \ell$, one has $\mathcal{Z}^{(1)}_{\varpi_1+\rho}, \ldots, \mathcal{Z}^{(1)}_{\varpi_p+\rho} \in Y_p$ and $\mathcal{Z}^{(1)}_{\varpi_1+\rho}, \ldots, \mathcal{Z}^{(1)}_{p\varpi_1+\rho} \in Y_p$. It is clear for $p = 1$. If it is true for $p-1$, then Y_p contains the following sums:

  $$\begin{aligned}
  \mathcal{Z}^{(1)}_{(p-1)\varpi_1+\rho} \cdot \mathcal{Z}^{(1)}_{\varpi_1+\rho} &= \mathcal{Z}^{(1)}_{p\varpi_1+\rho} + \mathcal{Z}^{(1)}_{(p-2)\varpi_1+\varpi_2+\rho}\\
  \mathcal{Z}^{(1)}_{(p-2)\varpi_1+\rho} \cdot \mathcal{Z}^{(1)}_{\varpi_2+\rho} &= \mathcal{Z}^{(1)}_{(p-2)\varpi_1+\varpi_2+\rho} + \mathcal{Z}^{(1)}_{(p-3)\varpi_1+\varpi_3+\rho}\\
  \mathcal{Z}^{(1)}_{\varpi_1+\rho} \cdot \mathcal{Z}^{(1)}_{\varpi_{p-1}+\rho} &= \mathcal{Z}^{(1)}_{\varpi_1+\varpi_{p-1}+\rho} + \mathcal{Z}^{(1)}_{\varpi_p+\rho}\\
  q^{d\cdot 2p\frac{\ell}{2}}\, \mathcal{Z}^{(p)}_{\varpi_1+\rho} &= \sum_{k=1}^{p} (-1)^{k+1}\, q^{d\cdot 2(\frac{\ell}{2}-k+1)}\, \mathcal{Z}^{(1)}_{(p-k)\varpi_1+\varpi_k+\rho}.
  \end{aligned}$$

  The elements $\mathcal{Z}^{(1)}_{p\varpi_1+\rho}, \mathcal{Z}^{(1)}_{(p-2)\varpi_1+\varpi_2+\rho}, \ldots, \mathcal{Z}^{(1)}_{\varpi_p+\rho}$ are combinations of these, because the determinant of the system is $(-1)^{p+1} q^{d\cdot(\ell-p)} \frac{q^{d\cdot p} - q^{-d\cdot p}}{q^d - q^{-d}} \neq 0$.

- Case B_ℓ ($\ell \ge 2$): the shortest representatives of elements in W/W_{ϖ_1} are given by $\{1, s_1, s_2s_1, \ldots, s_\ell\cdots s_1, s_{\ell-1}s_\ell s_{\ell-1}\cdots s_1, s_{\ell-2}s_{\ell-1}s_\ell s_{\ell-1}\cdots s_1, \ldots, s_1\cdots s_\ell\cdots s_1\}$. Let $1 \le p \le \ell$. For $1 \le k \le \ell$ and $w = s_{k-1}\cdots s_1$, one has:

– $p\varpi_1 + w^{-1}\rho$ belongs to a wall iff $k > p$, it is $(p-k)\varpi_1 + \varpi_k + \rho$ if $k \leq p$ and $k \leq \ell - 1$, and $2\varpi_\ell + \rho$ if $k = p = \ell$;
– $(\varpi_1 \mid w^{-1}\rho) = d(2\ell - 2k + 1)$ ($d = \frac{1}{2}$ in Bourbaki's normalization).

For $1 \leq k \leq \ell - 1$ and $w = s_k s_{k+1} \cdots s_\ell s_{\ell-1} \cdots s_1$, or $k = \ell$ and $w = s_\ell s_{\ell-1} \cdots s_1$, one has:

– $p\varpi_1 + w^{-1}\rho$ belongs to a wall except if p is odd and $k = \frac{2\ell-p+1}{2}$, in which case $p\varpi_1 + w^{-1}\rho = (s_{k-1}s_{k-2}\cdots s_1)^{-1}\rho$;
– $(\varpi_1 \mid w^{-1}\rho) = d(2k - 2\ell - 1)$.

We thus have for $1 \leq p \leq \ell - 1$, p odd:

$$q^{d\cdot 2p(2\ell-1)}\ \mathcal{Z}^{(p)}_{\varpi_1+\rho} = \textstyle\sum_{k=1}^{p}(-1)^{k+1}\ q^{d\cdot 2(2\ell-2k+1)}\ \mathcal{Z}^{(1)}_{(p-k)\varpi_1+\varpi_k+\rho}$$

and for $1 \leq p \leq \ell - 1$, p even:

$$q^{d\cdot 2p(2\ell-1)}\ \mathcal{Z}^{(p)}_{\varpi_1+\rho} = \textstyle\sum_{k=1}^{p}(-1)^{k+1}\ q^{d\cdot 2(2\ell-2k+1)}\ \mathcal{Z}^{(1)}_{(p-k)\varpi_1+\varpi_k+\rho} + q^{-d\cdot 2p}\ \mathcal{Z}^{(1)}_{\rho}$$

and similar relations for $p = \ell$, but with the term $(-1)^{\ell+1}q^{d\cdot 2}\ \mathcal{Z}^{(1)}_{2\varpi_\ell+\rho}$ in the sum for $k = \ell$. An induction similar to the case A_ℓ shows that Y is the $\mathbb{C}(q)$-subalgebra generated by the $\mathcal{Z}^{(1)}_{\varpi_1+\rho}, \mathcal{Z}^{(1)}_{\varpi_2+\rho}, \ldots, \mathcal{Z}^{(1)}_{2\varpi_l+\rho}$. One has to use the tensor product decomposition rules:

– $\mathrm{L}(m\varpi_1) \otimes \mathrm{L}(\varpi_n) = \mathrm{L}(m\varpi_1 + \varpi_n) \oplus \mathrm{L}((m-1)\varpi_1 + \varpi_{n+1}) \oplus \mathrm{Z}$ (for $1 \leq n \leq \ell - 2$, $1 \leq m$), where the summands in Z are some $\mathrm{L}(j\varpi_1 + \varpi_k)$ with $k \leq n$, $j + k \leq m + n - 2$;
– $\mathrm{L}(m\varpi_1) \otimes \mathrm{L}(\varpi_{\ell-1}) = \mathrm{L}(m\varpi_1 + \varpi_{\ell-1}) \oplus \mathrm{L}((m-1)\varpi_1 + 2\varpi_\ell) \oplus \mathrm{Z}$ (for $1 \leq m$), where the summands in Z are some $\mathrm{L}(j\varpi_1 + \varpi_k)$ with $k \leq \ell - 1$, $j + k \leq m + \ell - 3$;
– $\mathrm{L}(\varpi_\ell)^{\otimes 2}$ belongs to the subring generated in $\mathcal{R}$ by $\mathrm{L}(\varpi_1), \ldots, \mathrm{L}(\varpi_{\ell-1}), \mathrm{L}(2\varpi_\ell)$.

□

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