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ANOTHER PROOF OF JOSEPH AND LETZTER'S SEPARATION OF VARIABLES THEOREM FOR QUANTUM GROUPS

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Abstract. Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra and let G be the corresponding simply-connected algebraic group. A theorem of Kostant states that the universal enveloping algebra of $\mathfrak{g}$ is a free module over its center. A theorem of Richardson states that the algebra of regular functions on G is a free module over the subalgebra of regular class functions. Joseph and Letzter extended Kostant's theorem to the case of the quantized enveloping algebra of $\mathfrak{g}$. Using the theory of crystal bases as the main tool, we prove a quantum analogue of Richardson's theorem. From it, we recover Joseph and Letzter's result by a kind of "quantum duality principle".

1. Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra. The universal enveloping algebra $U(\mathfrak{g})$ is a $\mathfrak{g}$-module for the adjoint action, and this module, being the sum of its finite-dimensional submodules, is completely reducible. Let Z be the subalgebra of invariant elements in $U(\mathfrak{g})$, in other words Z is the center of $U(\mathfrak{g})$. Kostant [Ko] proved the existence of a $\mathfrak{g}$-submodule $\mathcal{K}$ of $U(\mathfrak{g})$ such that the multiplication in $U(\mathfrak{g})$ affords an isomorphism of $\mathfrak{g}$-modules from $Z \otimes_{\mathbb{C}} \mathcal{K}$ onto $U(\mathfrak{g})$.

The algebra $U(\mathfrak{g})$ has a q-analogue, usually called the quantized enveloping algebra of $\mathfrak{g}$ and denoted by $U_q(\mathfrak{g})$. (More precisely, we will choose the "simply-connected variant" of it, as explained below.) Being a Hopf algebra over the field $\mathbb{C}(q)$ of rational functions, $U_q(\mathfrak{g})$ is a left module over itself for the adjoint action. Joseph and Letzter defined $\mathcal{F}(U_q(\mathfrak{g}))$ to be the sum of the finite-dimensional submodules of $U_q(\mathfrak{g})$ and studied its properties (see [JL] and [Jo]). They observed that $\mathcal{F}(U_q(\mathfrak{g}))$ is a subalgebra of $U_q(\mathfrak{g})$ and a completely reducible $U_q(\mathfrak{g})$-module. Let Z_q be the subspace of invariants in $\mathcal{F}(U_q(\mathfrak{g}))$, i.e. the center of $U_q(\mathfrak{g})$. The following statement is an important

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result of [JL].

Theorem 1. *There exists a $U_q(\mathfrak{g})$-submodule $\mathcal{K}_q$ of $\mathcal{F}(U_q(\mathfrak{g}))$ such that*

- *the multiplication in the algebra $U_q(\mathfrak{g})$ affords an isomorphism of $U_q(\mathfrak{g})$-modules from $Z_q \otimes_{\mathbb{C}(q)} \mathcal{K}_q$ onto $\mathcal{F}(U_q(\mathfrak{g}))$;*
- *any simple finite-dimensional $U_q(\mathfrak{g})$-module has in $\mathcal{K}_q$ a multiplicity equal to the dimension of its zero-weight subspace.*

On the other hand, consider the simply-connected algebraic group G corresponding to $\mathfrak{g}$ and denote its function algebra by $\mathbb{C}[G]$. Acting on itself by conjugation, G acts also on $\mathbb{C}[G]$. Let $\mathbb{C}[G]^G$ be the subalgebra of invariant elements in $\mathbb{C}[G]$, i.e. the subalgebra of regular class functions. Richardson [Ri] proved the existence of a G-submodule $\mathcal{H}$ of $\mathbb{C}[G]$ such that the multiplication in $\mathbb{C}[G]$ affords an isomorphism of G-modules from $\mathbb{C}[G]^G \otimes_{\mathbb{C}} \mathcal{H}$ onto $\mathbb{C}[G]$.

Richardson's theorem has a q-analogue. Indeed, finite-dimensional $\mathfrak{g}$-modules may be deformed into $U_q(\mathfrak{g})$-modules, and the matrix coefficients of the $U_q(\mathfrak{g})$-modules obtained in this manner span a subalgebra $\mathbb{C}_q[G]$ in the Hopf dual algebra of $U_q(\mathfrak{g})$. This algebra $\mathbb{C}_q[G]$ is a $U_q(\mathfrak{g})$-module for the coadjoint action; let $\mathbb{C}_q[G]^G$ be its subalgebra of invariants.

Theorem 2. *There exists a $U_q(\mathfrak{g})$-submodule $\mathcal{H}_q$ of $\mathbb{C}_q[G]$ such that*

- *the multiplication in the algebra $\mathbb{C}_q[G]$ affords an isomorphism of $U_q(\mathfrak{g})$-modules from $\mathbb{C}_q[G]^G \otimes_{\mathbb{C}(q)} \mathcal{H}_q$ onto $\mathbb{C}_q[G]$;*
- *any simple finite-dimensional $U_q(\mathfrak{g})$-module has in $\mathcal{H}_q$ a multiplicity equal to the dimension of its zero-weight subspace.*

The quantum situation seems simpler than the classical one in two respects. On the one hand, whereas there is no general relation between Kostant's and Richardson's results, Theorems 1 and 2 are equivalent statements. This fact follows from the existence, first noticed by Caldero [Ca1], of a canonical isomorphism between the $U_q(\mathfrak{g})$-modules $\mathbb{C}_q[G]$ and $\mathcal{F}(U_q(\mathfrak{g}))$ which preserves partially the multiplication (see Theorem 3). On the other hand, Kostant's and Richardson's proofs rely on a geometric analysis of the G-varieties $\mathfrak{g}$ and G, while the proof of Theorem 1 by Joseph and Letzter uses only algebra.

In this note, we present another proof of Joseph and Letzter's theorem. We actually prove Theorem 2, because things seem more natural on this side. Our approach is based on Kashiwara's theory of crystal bases. The algebra $\mathbb{C}_q[G]$ has a natural crystal basis in which we define explicitly a suitable $U_q(\mathfrak{g})$-submodule $\mathcal{H}_q$. The nice behaviour of crystal bases under tensor product allows us to show that the properties stated in Theorem 2 hold for this particular choice of $\mathcal{H}_q$. By contrast, Joseph and Letzter allowed more freedom in the choice of $\mathcal{K}_q$, but they needed several auxiliary results to conclude (namely Corollary 7.3.3, Lemma 7.3.4, and Proposition 7.3.7 in [Jo]), which in turn required separate and unrelated proofs.

The plan of this paper is the following. In Sections 2.2 and 2.3, we recall the definition of the quantized enveloping algebra $U_q(\mathfrak{g})$ and basic facts concerning its representation theory. In Section 2.4, following an idea which goes back to Reshetikhin and Semenov-Tian-Shansky [RS], we use the universal R-matrix of $U_q(\mathfrak{g})$ to construct an isomorphism between $\mathbb{C}_q[G]$ and $\mathcal{F}(U_q(\mathfrak{g}))$, which leads to the equivalence between Theorems 1 and 2. In Sections 2.5 and 2.6, we recall the definition and the fundamental properties of crystal bases of finite-dimensional $U_q(\mathfrak{g})$-modules. In Section 3.1, we sketch our proof to Theorem 2. The final Sections 3.2–3.5 contain all the details of the proof.

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2. Definitions and auxiliary facts

2.1. Basic conventions

By convention, all modules considered in this paper are left modules.

We fix a ground field k. (From Section 2.2 to the end of the paper, k will be the field $\mathbb{C}(q)$ of rational functions.) The tensor products are taken over k, except where otherwise stated. We denote the dual of a k-vector space V by $V^* = \mathrm{Hom}_k(V, k)$.

For any Hopf algebra H over k, the comultiplication map, the augmentation map, and the antipode will be denoted by

$$\Delta_H : H \to H \otimes_k H, \qquad \varepsilon_H : H \to k, \qquad \text{and} \quad S_H : H \to H.$$

When the context makes clear what H is, and especially when H is the quantized enveloping algebra $U_q(\mathfrak{g})$, we will abbreviate the notation in Δ, ε, and S. We will also use the Sweedler notation, writing

$$\Delta_H(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$

for any $x \in H$.

Let H be a Hopf algebra over k. For any H-modules M and N, the rule

$$x \cdot (m \otimes n) = \sum_{(x)} (x_{(1)} \cdot m) \otimes (x_{(2)} \cdot n)$$

where $x \in H$, $m \in M$, and $n \in N$, endows the tensor product $M \otimes_k N$ with an H-module structure. For any H-module M, the dual space M^* becomes an H-module when endowed with the action

$$x \cdot m^* = \left(M \to k,\ m \mapsto \langle m^*, S_H(x) \cdot m\rangle\right),$$

where $x \in H$ and $m^* \in M^*$. The subspace of invariants of an H-module M is

$$M^H = \{m \in M \mid \forall x \in H,\ x \cdot m = \varepsilon_H(x)m\}.$$

The space H is a module over itself for the adjoint action, defined by the formula

$$x \cdot y = \sum_{(x)} x_{(1)} y\, S_H(x_{(2)})$$

where $x, y \in H$. The dual space H^* is an H-module for the coadjoint action, defined by

$$x \cdot \varphi = \big(H \to k,\ y \mapsto \varphi(S_H(x_{(1)}) y x_{(2)})\big)$$

where $x \in H$ and $\varphi \in H^*$. It is also an associative algebra for the multiplication

$$\varphi\psi = \Big(H \to k,\ y \mapsto \sum_{(y)} \varphi(y_{(1)})\, \psi(y_{(2)})\Big),$$

where $\varphi, \psi \in H^*$, with ε_H as unit.

Let M be an H-module. To any pair $(m, m^*) \in M \times M^*$, we associate the matrix coefficient

$$c^M_{m^*,m} : (H \to k,\ x \mapsto \langle m^*, x \cdot m\rangle)\,.$$

Endowing H^* with the coadjoint action, the map

$$\big(M^* \otimes M \to H^*,\ m^* \otimes m \mapsto c^M_{m^*,m}\big)$$

is a homomorphism of H-modules. The subspace spanned in H^* by the matrix coefficients of the finite-dimensional H-modules is called the restricted dual of H; since the duals of the structure maps of H endow this space with the structure of a Hopf algebra over k (see Section 6.0 of [Sw]), it is also called the Hopf dual algebra of H.

2.2. The quantized enveloping algebra $U_q(\mathfrak{g})$

The choice of a Cartan subalgebra in $\mathfrak{g}$ yields a root system. Let P be the weight lattice, let $\{\alpha_i \mid i \in I\}$ be a set of simple roots, let $(\varpi_i)_{i\in I}$ be the corresponding family of fundamental weights, let $Q_+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\, \alpha_i$ be the positive cone, and let $P_{++} = \sum_{i\in I} \mathbb{Z}_{\geq 0}\, \varpi_i$ be the cone of dominant integral weights. Up to normalization, there is a unique scalar product on $P \otimes_{\mathbb{Z}} \mathbb{R}$ such that the Cartan matrix of $\mathfrak{g}$ has $a_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}$ for coefficients. We may impose the additional requirement that $(P|P) \subseteq \mathbb{Z}$. We then define

$d_i = \frac{1}{2}(\alpha_i|\alpha_i) \in \mathbb{Z}$ and $\alpha_i^\vee = \alpha_i/d_i$. We finally denote the longest element in the Weyl group by w_0.

From now on, the ground field k is the field $\mathbb{C}(q)$ of rational functions. For any natural integer n and any index $i \in I$, we set

$$[n]_i = \frac{q^{d_i n} - q^{-d_i n}}{q^{d_i} - q^{-d_i}} \qquad \text{and} \qquad [n]_i! = \prod_{r=1}^{n} [r]_i.$$

The "simply-connected" variant of the quantized enveloping algebra of $\mathfrak{g}$ is the $\mathbb{C}(q)$-algebra $U_q(\mathfrak{g})$ generated by elements $(K_\lambda)_{\lambda\in P}$, $(E_i)_{i\in I}$, and $(F_i)_{i\in I}$, with the relations

$$K_\lambda\, E_i = q^{(\lambda|\alpha_i)} E_i\, K_\lambda, \qquad K_\lambda\, F_i = q^{-(\lambda|\alpha_i)} F_i\, K_\lambda,$$
$$K_\lambda\, K_\mu = K_{\lambda+\mu}, \qquad [E_i, F_j] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}},$$
$$\sum_{\substack{r,s\geq 0 \\ r+s=1-\langle\alpha_i^\vee,\alpha_j\rangle}} \frac{[r+s]_i!}{[r]_i![s]_i!}\, E_i^r\, E_j\, E_i^s = 0 \qquad (\text{when } i \neq j),$$
$$\sum_{\substack{r,s\geq 0 \\ r+s=1-\langle\alpha_i^\vee,\alpha_j\rangle}} \frac{[r+s]_i!}{[r]_i![s]_i!}\, F_i^r\, F_j\, F_i^s = 0 \qquad (\text{when } i \neq j).$$

This definition appeared in [JL] and in Section (0.3) of [DKP].

There are unique morphisms of algebras

$$\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes_{\mathbb{C}(q)} U_q(\mathfrak{g}) \qquad \text{and} \qquad \varepsilon : U_q(\mathfrak{g}) \to \mathbb{C}(q)$$

such that

$$\Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \qquad \varepsilon(K_\lambda) = 1,$$
$$\Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i, \qquad \varepsilon(E_i) = 0,$$
$$\Delta(F_i) = F_i \otimes K_{-\alpha_i} + 1 \otimes F_i, \qquad \varepsilon(F_i) = 0.$$

Endowed with this coproduct Δ and this augmentation ε, the space $U_q(\mathfrak{g})$ becomes a Hopf algebra.

2.3. $U_q(\mathfrak{g})$-modules

For any $\mu \in P$, we define the μ-weight subspace of a $U_q(\mathfrak{g})$-module M by

$$M_\mu = \{m \in M \mid \forall \lambda \in P, \quad K_\lambda \cdot m = q^{(\lambda|\mu)} m\}.$$

A module M is said to be integrable if it is the sum of both its finite-dimensional submodules and its weight subspaces. It is known that integrable modules are completely reducible. In the sequel, we will deal only with integrable $U_q(\mathfrak{g})$-modules.

Given a dominant weight $\lambda \in P_{++}$, there is a unique up to isomorphism simple integrable $U_q(\mathfrak{g})$-module that has λ as highest weight: we denote it by $V(\lambda)$. Let $C(\lambda)$ be the linear span of the matrix coefficients of the module $V(\lambda)$; by the results recalled in Section 2.1, the subspaces $C(\lambda)$ are submodules of the coadjoint $U_q(\mathfrak{g})$-module $(U_q(\mathfrak{g}))^*$.

The subspace spanned in the dual of $U_q(\mathfrak{g})$ by the matrix coefficients of the integrable $U_q(\mathfrak{g})$-modules is

$$\mathbb{C}_q[G] = \bigoplus_{\lambda \in P_{++}} C(\lambda). \tag{1}$$

Being a submodule of $(U_q(\mathfrak{g}))^*$, this space $\mathbb{C}_q[G]$ is a $U_q(\mathfrak{g})$-module for the coadjoint action. It is also a Hopf subalgebra of the Hopf dual algebra of $U_q(\mathfrak{g})$, since the category of finite-dimensional integrable $U_q(\mathfrak{g})$-modules is closed under tensor product and dualization. By Lemma 7.1.9 in [Jo], the space $\mathbb{C}_q[G]$ separates the points of $U_q(\mathfrak{g})$, so the canonical duality between $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$ is non-degenerate.

We denote by $\mathbb{C}_q[G]^G$ and Z_q the subspaces of invariants in the $U_q(\mathfrak{g})$-modules $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$, the latter being endowed with the adjoint action; thus Z_q is the center of $U_q(\mathfrak{g})$. As mentioned in Section 1, we let $\mathcal{F}(U_q(\mathfrak{g}))$ denote the sum of the finite-dimensional submodules of the adjoint module $U_q(\mathfrak{g})$. It is easily seen that the $U_q(\mathfrak{g})$-module $\mathcal{F}(U_q(\mathfrak{g}))$ is integrable and is a subalgebra of $U_q(\mathfrak{g})$ which contains Z_q. The next lemma shows that $\mathbb{C}_q[G]^G$ is a subalgebra of $\mathbb{C}_q[G]$.

Lemma 1. *For any invariant element $\varphi \in \mathbb{C}_q[G]^G$, the multiplication map*

$$(\mathbb{C}_q[G] \to \mathbb{C}_q[G],\ \psi \mapsto \varphi\psi)$$

is $U_q(\mathfrak{g})$-linear. The subspace $\mathbb{C}_q[G]^G$ is a subalgebra of $\mathbb{C}_q[G]$.

Proof. For any $\varphi \in \mathbb{C}_q[G]^G$, $\psi \in \mathbb{C}_q[G]$, and $x \in U_q(\mathfrak{g})$, we have

$$\begin{aligned} x \cdot (\varphi\psi) &= \begin{pmatrix} U_q(\mathfrak{g}) \to \mathbb{C}(q) \\ y \mapsto \sum_{(x)} \langle \varphi\psi, S(x_{(1)}) y x_{(2)} \rangle \end{pmatrix} \\ &= \Big(y \mapsto \sum_{(x),(y)} \langle \varphi, S(x_{(2)}) y_{(1)} x_{(3)} \rangle \ \langle \psi, S(x_{(1)}) y_{(2)} x_{(4)} \rangle \Big) \\ &= \Big(y \mapsto \sum_{(x),(y)} \langle \varphi, y_{(1)} \rangle \ \langle \psi, S(x_{(1)}) y_{(2)} x_{(2)} \rangle \Big) \\ &= \varphi\ (x \cdot \psi). \end{aligned}$$

The second and fourth equalities come from the definition of the product in $\mathbb{C}_q[G]$ and the third one comes from the $U_q(\mathfrak{g})$-invariance of φ. This computation proves the first assertion. Being $U_q(\mathfrak{g})$-linear, the left multiplication by an invariant element φ sends the space of invariants $\mathbb{C}_q[G]^G$ into itself, which implies the second assertion. $\square$

2.4. The R-matrix and the equivalence between Theorems 1 and 2 The R-matrix of $U_q(\mathfrak{g})$ is an element of $U_q(\mathfrak{g})\widehat{\otimes}_{\mathbb{C}(q)}U_q(\mathfrak{g})$ given by a certain sum $R_{12} = \sum_j a_j \otimes b_j$, where the elements a_j (respectively, b_j) belong to the subalgebra generated by the F_i and the K_λ (respectively, by the E_i and the K_λ). Although the sum is infinite, the special structure of R_{12} (see Section 13 of [Dr1]) makes it define two maps

$$l^+ : \left(\mathbb{C}_q[G] \to U_q(\mathfrak{g}),\ \varphi \mapsto \langle \mathrm{id}_{U_q(\mathfrak{g})} \otimes \varphi, R_{12}\rangle\right),$$

$$l^- : \left(\mathbb{C}_q[G] \to U_q(\mathfrak{g}),\ \varphi \mapsto \langle (\varphi \circ S_{U_q(\mathfrak{g})}) \otimes \mathrm{id}_{U_q(\mathfrak{g})}, R_{12}\rangle\right).$$

Note that this precise point requires the use of the simply-connected variant of the quantized enveloping algebra.

Reshetikhin and Semenov-Tian-Shansky [RS] have used the maps l^+ and l^- to define a third map

$$I : \Big(\mathbb{C}_q[G] \to U_q(\mathfrak{g}),\ \varphi \mapsto \sum_{(\varphi)} l^+(\varphi_{(1)})\ S(l^-(\varphi_{(2)}))\Big).$$

With the notation $R_{21} = \sum_j b_j \otimes a_j$, one may also write

$$I(\varphi) = \langle \varphi \otimes \mathrm{id}_{U_q(\mathfrak{g})}, R_{21}R_{12}\rangle.$$

The following result is more or less well-known (see Proposition 3.3 in [Dr2], Proposition 2.1 in [Ma], [Ca1], and Proposition 7.1.23 in [Jo]).

Theorem 3. *(i) The map I affords an isomorphism of $U_q(\mathfrak{g})$-modules from $\mathbb{C}_q[G]$ onto $\mathcal{F}(U_q(\mathfrak{g}))$.*
(ii) For any $\varphi \in \mathbb{C}_q[G]^G$ and $\psi \in \mathbb{C}_q[G]$, one has $I(\varphi\psi) = I(\varphi)I(\psi)$.
(iii) The map I induces an isomorphism of algebras from $\mathbb{C}_q[G]^G$ onto Z_q.

Proof. The element $R_{21}R_{12}$ in $U_q(\mathfrak{g})\widehat{\otimes}_{\mathbb{C}(q)}U_q(\mathfrak{g})$ commutes with all the elements of the form $\Delta(y)$, where $y \in U_q(\mathfrak{g})$. Consequently for any $\varphi, \psi \in \mathbb{C}_q[G]$ and $x \in U_q(\mathfrak{g})$

$$\begin{aligned}
\langle \psi, I(x\cdot\varphi)\rangle &= \langle (x\cdot\varphi)\otimes\psi, R_{21}R_{12}\rangle \\
&= \sum_{(x)} \langle \varphi\otimes\psi, (S(x_{(1)})\otimes 1)(R_{21}R_{12})(x_{(2)}\otimes 1)\rangle \\
&= \sum_{(x)} \langle \varphi\otimes\psi, (S(x_{(1)})\otimes 1)(R_{21}R_{12})\Delta(x_{(2)})(1\otimes S(x_{(3)}))\rangle \\
&= \sum_{(x)} \langle \varphi\otimes\psi, (S(x_{(1)})\otimes 1)\Delta(x_{(2)})(R_{21}R_{12})(1\otimes S(x_{(3)}))\rangle \\
&= \sum_{(x)} \langle \varphi\otimes\psi, (1\otimes x_{(1)})(R_{21}R_{12})(1\otimes S(x_{(2)}))\rangle \\
&= \sum_{(x)} \langle \psi, x_{(1)} I(\varphi) S(x_{(2)})\rangle \\
&= \langle \psi, x\cdot I(\varphi)\rangle.
\end{aligned}$$

Thus I is $U_q(\mathfrak{g})$-linear. (Despite the infinite sum in the R-matrix, no trouble arises here, because the computation can indeed be done using only the well-defined maps l^+ and l^-; see the proof of Proposition 3 in [BS].)

The definition of an R-matrix implies that $l^\pm$ are morphisms of coalgebras and antihomomorphisms of algebras (see Section 10 of [Dr1]). Then $l^\pm = S_{U_q(\mathfrak{g})} \circ l^\pm \circ S_{\mathbb{C}_q[G]}$, and thus for any $\varphi, \psi \in \mathbb{C}_q[G]$ we have

$$\begin{aligned}
&I\Big(\sum_{(\psi)}\big([(l^+ \circ S_{\mathbb{C}_q[G]})(\psi_{(1)})] \cdot \varphi\big)\psi_{(2)}\Big) \\
&\quad = \sum_{(\psi)} \sum_{\left([(l^+\circ S_{\mathbb{C}_q[G]})(\psi_{(1)})]\cdot\varphi\right)} l^+\big(([(l^+ \circ S_{\mathbb{C}_q[G]})(\psi_{(1)})] \cdot \varphi)_{(1)}\psi_{(2)}\big) \\
&\qquad\qquad \times (S_{U_q(\mathfrak{g})} \circ l^-)\big(([(l^+ \circ S_{\mathbb{C}_q[G]})(\psi_{(1)})] \cdot \varphi)_{(2)}\psi_{(3)}\big) \\
&\quad = \sum_{(\psi)} l^+(\psi_{(2)})\; I\big([(l^+ \circ S_{\mathbb{C}_q[G]})(\psi_{(1)})] \cdot \varphi\big)\; (S_{U_q(\mathfrak{g})} \circ l^-)(\psi_{(3)}) \\
&\quad = \sum_{(\psi)} l^+(\psi_{(2)})\; \big([(l^+ \circ S_{\mathbb{C}_q[G]})(\psi_{(1)})] \cdot I(\varphi)\big)\; (S_{U_q(\mathfrak{g})} \circ l^-)(\psi_{(3)}) \\
&\quad = \sum_{(\psi)} l^+(\psi_{(3)})\; (l^+ \circ S_{\mathbb{C}_q[G]})(\psi_{(2)})\; I(\varphi) \\
&\qquad\qquad \times (S_{U_q(\mathfrak{g})} \circ l^+ \circ S_{\mathbb{C}_q[G]})(\psi_{(1)})\; (S_{U_q(\mathfrak{g})} \circ l^-)(\psi_{(4)}) \\
&\quad = \sum_{(\psi)} l^+\big(S_{\mathbb{C}_q[G]}(\psi_{(2)})\psi_{(3)}\big)\; I(\varphi)\; l^+(\psi_{(1)})\; (S_{U_q(\mathfrak{g})} \circ l^-)(\psi_{(4)}) \\
&\quad = I(\varphi)I(\psi).
\end{aligned} \tag{2}$$

We also have

$$I(\varepsilon_U) = 1. \tag{3}$$

Finally the known relation $(S_{U_q(\mathfrak{g})} \otimes S_{U_q(\mathfrak{g})})(R_{12}) = R_{12}$ (see Proposition 3.1 in [Dr2]) implies that for any $\varphi, \psi \in \mathbb{C}_q[G]$,

$$\begin{aligned}
\langle I(\varphi), \psi\rangle &= \langle \varphi \otimes \psi, R_{21}R_{12}\rangle \\
&= \langle \varphi \otimes \psi, (S_{U_q(\mathfrak{g})} \otimes S_{U_q(\mathfrak{g})})(R_{12}R_{21})\rangle \\
&= \langle S_{\mathbb{C}_q[G]}(\psi) \otimes S_{\mathbb{C}_q[G]}(\varphi), R_{21}R_{12}\rangle \\
&= \langle I(S_{\mathbb{C}_q[G]}(\psi)), S_{\mathbb{C}_q[G]}(\varphi)\rangle.
\end{aligned} \tag{4}$$

Let $\lambda \in P_{++}$ and choose a lowest weight vector m_{lw} in $V(\lambda)$ and a highest weight vector m^*_{hw} in $V(\lambda)^*$ such that $\langle m^*_{\mathrm{hw}}, m_{\mathrm{lw}}\rangle = 1$. The description of the R-matrix given in Section 13 of [Dr1] implies that

$$l^+(c^{V(\lambda)}_{m^*_{\mathrm{hw}},m}) = \langle m^*_{\mathrm{hw}}, m\rangle\, K_{w_0\lambda} \quad \text{and} \quad l^-(c^{V(\lambda)}_{m^*,m_{\mathrm{lw}}}) = \langle m^*, m_{\mathrm{lw}}\rangle\, K_{-w_0\lambda},$$

for any $m \in M$ and $m^* \in M^*$. Taking a basis (m_k) of $V(\lambda)$ and the dual basis (m_k^*) of $V(\lambda)^*$, we therefore get

$$\begin{aligned} I\left(c^{V(\lambda)}_{m^*_{\mathrm{hw}},m_{\mathrm{lw}}}\right) &= \sum_k l^+\left(c^{V(\lambda)}_{m^*_{\mathrm{hw}},m_k}\right)\,(S\circ l^-)\left(c^{V(\lambda)}_{m^*_k,m_{\mathrm{lw}}}\right) \\ &= \sum_k \langle m^*_{\mathrm{hw}}, m_k\rangle\langle m^*_k, m_{\mathrm{lw}}\rangle\, K_{2w_0\lambda} \\ &= K_{2w_0\lambda}, \end{aligned}$$

and so $K_{2w_0\lambda}$ belongs to the image of I.

Formulas (2) and (3) show that the image of I is a subalgebra of $U_q(\mathfrak{g})$. The $U_q(\mathfrak{g})$-linearity of I shows that (im I) is stable under the adjoint action of $U_q(\mathfrak{g})$. From the relations

$$\begin{aligned} E_i \cdot K_{-2\varpi_i} &= (q^{(\alpha_i|\alpha_i)} - 1)\, K_{-2\varpi_i}\, E_i, \\ F_i \cdot K_{-2\varpi_i} &= (1 - q^{(\alpha_i|\alpha_i)})\, F_i\, K_{\alpha_i - 2\varpi_i}, \end{aligned}$$

it follows that (im I) is a subalgebra of $U_q(\mathfrak{g})$ which contains all the elements $K_{-2\lambda}$, $K_{-2\varpi_i}\, E_i$, and $F_i\, K_{\alpha_i-2\varpi_i}$, where $\lambda \in P_{++}$ and $i \in I$. This is enough to ensure that the $U_q(\mathfrak{g})$-modules $V(\mu)$, where $\mu \in P_{++}$, are pairwise non-isomorphic absolutely simple (im I)-modules. In other words, the duality between $\mathbb{C}_q[G] \subseteq (U_q(\mathfrak{g}))^*$ and (im I) $\subseteq U_q(\mathfrak{g})$ is non-degenerate. This non-degeneracy, the bijectivity of $S_{\mathbb{C}_q[G]}$, and Formula (4) prove all together the injectivity of I.

Since $\mathbb{C}_q[G]$ is the sum of its finite-dimensional $U_q(\mathfrak{g})$-submodules, the inclusion (im I) $\subseteq \mathcal{F}(U_q(\mathfrak{g}))$ holds. The proof of the equality (im I) $= \mathcal{F}(U_q(\mathfrak{g}))$ requires the fact, due to Joseph and Letzter, that the $U_q(\mathfrak{g})$-module $\mathcal{F}(U_q(\mathfrak{g}))$ is generated by the family $(K_{-2\lambda})_{\lambda\in P_{++}}$ (see [Ca1] for a simple proof of this). The assertions of Statement (i) are at last proved.

Statement (ii) is a particular case of Formula (2) above. Finally Statement (iii) follows from Statements (i) and (ii) and from Formula (3). □

According to Reshetikhin and Semenov-Tian-Shansky, the map I is a quantum analogue of the Killing isomorphism from $S(\mathfrak{g}^*)$ onto $S(\mathfrak{g})$. One can understand this by looking at the expansion of the R-matrix in powers of $(q-1)$ (see Theorem 3.4.2 in [Ro] for instance): letting $C \in \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g}$ be the Casimir element and identifying in the limit $q \to 1$ the subspace $\mathfrak{g} \subseteq U(\mathfrak{g})$ with a subspace of $U_q(\mathfrak{g})$, one has

$$R_{21}R_{12} = 1 \otimes 1 + (q-1)\, C + \cdots .$$

2.5. Crystals

Kashiwara's theory of crystals allows one to reduce certain problems in representation theory to simple combinatorics. For the convenience of the reader, we recall a few facts about this theory. Since our purposes only require the most basic part of the theory, we slightly simplified the terminology presented in the survey paper [Ka3]: for us, any crystal is semi-normal and any morphism is strict.

The set I is fixed as in Section 2.2. A crystal is a set B with maps

$$\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\},$$

where $i \in I$, such that:

- for any $i \in I$, one has $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$;
- for any $i \in I$ and $b \in B$, there is an integer $n > 0$ such that $\tilde{e}_i^n(b) = \tilde{f}_i^n(b) = 0$;
- for any $i \in I$ and $b, b' \in B$, the equalities $b' = \tilde{f}_i(b)$ and $b = \tilde{e}_i(b')$ are equivalent.

Given two crystals B_1 and B_2, a morphism from B_1 to B_2 is a map $g : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ such that $g(0) = 0$ and which commutes with the action of the operators $\tilde{e}_i$ and $\tilde{f}_i$. Then the crystals and their morphisms form a category.

For an element b of a crystal B, one sets

$$\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n(b) \neq 0\},$$

$$\varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n(b) \neq 0\}.$$

If $g : B_1 \to B_2$ is a morphism of crystals, then given $b \in B_1$ such that $g(b) \neq 0$, one has $\varepsilon_i(g(b)) = \varepsilon_i(b)$ and $\varphi_i(g(b)) = \varphi_i(b)$ for any $i \in I$.

Given a crystal B, we define a map $\mathrm{wt} : B \to P$ by letting the weight of an element $b \in B$ be

$$\mathrm{wt}(b) = \sum_{i \in I} (\varphi_i(b) - \varepsilon_i(b))\, \varpi_i.$$

If $g : B_1 \to B_2$ is a morphism of crystals, then for any $b \in B_1$ such that $g(b) \neq 0$, one has $\mathrm{wt}(g(b)) = \mathrm{wt}(b)$.

Given two crystals B_1 and B_2, one defines their direct sum $B_1 \oplus B_2$ and their tensor product $B_1 \otimes B_2$ by the following rules:

- the underlying set of $B_1 \oplus B_2$ is $B_1 \sqcup B_2$, and one glues the maps $\tilde{e}_i, \tilde{f}_i : B_1 \to B_1 \sqcup \{0\}$ and $\tilde{e}_i, \tilde{f}_i : B_2 \to B_2 \sqcup \{0\}$ to form the maps $\tilde{e}_i, \tilde{f}_i : B_1 \sqcup B_2 \to B_1 \sqcup B_2 \sqcup \{0\}$;
- the underlying set of $B_1 \otimes B_2$ is $B_1 \times B_2$, but one writes $b_1 \otimes b_2$ instead of (b_1, b_2);
- the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B_1 \otimes B_2$ are given by the rules

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{5}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \tag{6}$$

where one agrees that $b_1 \otimes 0 = 0 \otimes b_2 = 0$.

These operations are well-defined (i.e., afford crystals) and are associative (see Lemma 2.2.4 in [KK]). One also has $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$ (Formula (2.2.19) in [KK]).

A crystal B may be regarded as a graph B with oriented and coloured by I edges. The vertices of B are the elements of B, and one draws an arrow of colour i from a vertex b to a vertex b' if and only if $b' = \tilde{f}_i(b)$. Thus for instance, the graph defined by the direct sum of two crystals B_1 and B_2 is the disjoint union of the graphs defined by B_1 and B_2. With this graphical interpretation, one can carry to crystals the notions of connectedness and connected component.

Finally, given a crystal B, we say that an element $b \in B$ is a highest (respectively, lowest) weight vector if $\tilde{e}_i(b) = 0$ (respectively, $\tilde{f}_i(b) = 0$) for each $i \in I$.

2.6. Crystal bases

Let us go back to the representation theory of $U_q(\mathfrak{g})$. Kashiwara defined operators $\tilde{e}_i$ and $\tilde{f}_i$ that act on every integrable $U_q(\mathfrak{g})$-module. Their definition is as follows. Given an integrable module M and an index i, any vector m of weight λ can be written in a unique way as a finite sum

$$m = \sum_{n \geq 0} F_i^n \cdot m_n,$$

where $\lambda \in P$ and $m_n \in M_{\lambda + n\alpha_i}$ is such that $E_i \cdot m_n = 0$. One defines then

$$\tilde{e}_i(m) = \sum_{n \geq 1} [n]_i \; q^{-(\alpha_i|\lambda)} \; F_i^{n-1} \cdot m_n,$$

$$\tilde{f}_i(m) = \sum_{n \geq 0} \frac{1}{[n+1]_i} q^{(\alpha_i|\lambda - \alpha_i)} \; F_i^{n+1} \cdot m_n.$$

(These normalizations are necessary to ensure the compatibility between Kashiwara's notations and ours.)

Let $A \subseteq \mathbb{C}(q)$ be the subring of rational functions without pole at $q = 0$. A crystal basis of an integrable finite-dimensional $U_q(\mathfrak{g})$-module M is a pair (L, B) such that:

- L is an A-lattice in M and B is a basis of the $\mathbb{C}$-vector space L/qL;
- for any weight $\mu \in P$, the subspace $L_\mu = L \cap M_\mu$ is an A-lattice in M_μ and $B_\mu = B \cap (L_\mu / qL_\mu)$ is a basis of the $\mathbb{C}$-vector space L_μ/qL_μ;
- Kashiwara's operators $\tilde{e}_i$ and $\tilde{f}_i$ leave L stable and induce on L/qL operators (still denoted by $\tilde{e}_i$ and $\tilde{f}_i$) which leave $B \sqcup \{0\}$ stable;
- for any $i \in I$ and $b, b' \in B$, the equalities $b' = \tilde{f}_i(b)$ and $b = \tilde{e}_i(b')$ are equivalent.

A crystal basis (L, B) of an integrable $U_q(\mathfrak{g})$-module affords the crystal B. Moreover, Formula (2.4.2) in [Ka1] shows that the two notions of weight are compatible: if $b \in B_\mu$, then $\mathrm{wt}(b) = \mu$. Let M_1 and M_2 be two $U_q(\mathfrak{g})$-modules, with given crystal bases (L_1, B_1) and (L_2, B_2). A $U_q(\mathfrak{g})$-linear map $f : M_1 \to M_2$ will be called compatible with the crystal bases if f sends L_1 into L_2 and induces a map $\bar{f} : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ after reduction modulo q. Then $\bar{f}$ is a morphism of crystals. A crystal arising from a crystal basis of an integrable $U_q(\mathfrak{g})$-module will be called normal, as in Section 7 of [Ka3].

Let M_1 and M_2 be two modules, given with crystal bases (L_1, B_1) and (L_2, B_2). Denoting by $B_1 \oplus B_2$ the set of vectors in $L_1/qL_1 \oplus L_2/qL_2 \simeq (L_1 \oplus L_2)/q(L_1 \oplus L_2)$ of the form $b_1 \oplus 0$ or $0 \oplus b_2$, where $b_1 \in B_1$ and $b_2 \in B_2$, the pair $(L_1 \oplus L_2, B_1 \oplus B_2)$ is a crystal basis of the module $M_1 \oplus M_2$. Denoting by $B_1 \otimes B_2$ the set of elements $b_1 \otimes b_2 \in (L_1/qL_1) \otimes_{\mathbb{C}} (L_2/qL_2) \simeq (L_1 \otimes_A L_2)/q(L_1 \otimes_A L_2)$, where (b_1, b_2) runs over $B_1 \times B_2$, Theorem 1 in [Ka1] asserts that $(L_1 \otimes_A L_2, B_1 \otimes B_2)$ is a crystal basis of the module $M_1 \otimes_{\mathbb{C}(q)} M_2$. The crystals associated to these crystal bases are $B_1 \oplus B_2$ and $B_1 \otimes B_2$, respectively, as defined in Section 2.5. These constructions extend to a finite number of terms or factors.

Given a highest weight vector u_λ in the module $V(\lambda)$, there is a unique crystal basis $(L(\lambda), B(\lambda))$ of $V(\lambda)$ such that $L(\lambda)_\lambda = A\, u_\lambda$ and $B(\lambda)_\lambda = \{u_\lambda \bmod qL(\lambda)\}$ (see Theorems 2 and 3 in [Ka1]). We denote the residual class of u_λ in $B(\lambda)_\lambda$ by $\bar{u}_\lambda$ and take a representative $d_\lambda \in L(\lambda)_{w_0\lambda}$ of the unique element $\bar{d}_\lambda$ in $B(\lambda)_{w_0\lambda}$. The construction given in [Ka1] shows that the crystals $B(\lambda)$ are connected.

Theorem 3 in [Ka1] states that whenever two crystal bases (L_1, B_1) and (L_2, B_2) of two isomorphic integrable $U_q(\mathfrak{g})$-modules M_1 and M_2 are given, there is an isomorphism f from M_1 onto M_2 which carries (L_1, B_1) onto (L_2, B_2). The following proposition is a corollary of this uniqueness result.

Proposition 1. *(i) Let M be an integrable $U_q(\mathfrak{g})$-module and denote by $[M : V(\lambda)]$ the multiplicity of the simple module $V(\lambda)$ in M. For any crystal basis (L, B) of M, there is an isomorphism from $\bigoplus_\lambda V(\lambda)^{\oplus [M:V(\lambda)]}$ onto M that carries the crystal basis $\bigoplus_\lambda (L(\lambda), B(\lambda))^{\oplus [M:V(\lambda)]}$ onto (L, B).*
(ii) Let M be an integrable $U_q(\mathfrak{g})$-module, let (L, B) be a crystal basis of M, and let N be an isotypical component of M. If one sets $L_N = L \cap N$ and $B_N = B \cap (L_N/qL_N)$, then (L_N, B_N) is a crystal basis of N.
(iii) Let $\mu, \nu \in P_{++}$. The $U_q(\mathfrak{g})$-linear map $p_{\mu,\nu}$ from $V(\mu) \otimes_{\mathbb{C}(q)} V(\nu)$ to $V(\mu + \nu)$ that sends $u_\mu \otimes u_\nu$ to $u_{\mu+\nu}$ is compatible with the crystal bases $(L(\mu) \otimes_A L(\nu), B(\mu) \otimes B(\nu))$ and $(L(\mu + \nu), B(\mu + \nu))$.
(iv) A crystal B is normal if and only if each of its connected components is isomorphic to a crystal $B(\lambda)$, for some $\lambda \in P_{++}$. If such a crystal B comes from a module M, then the number of connected components of B that are isomorphic to a given $B(\lambda)$ is equal to the multiplicity of $V(\lambda)$ in M.

To clarify the discussion in the next sections, we collect in a proposition several known results. For an element b of a crystal, we put $\varepsilon(b) = \sum_i \varepsilon_i(b)\varpi_i$.

Proposition 2. *(i) The crystal $B(\lambda)$ has exactly one highest weight vector and one lowest weight vector, namely $\bar{u}_\lambda$ and $\bar{d}_\lambda$, respectively.*
(ii) Let B_1 and B_2 be normal crystals, and let $b_1 \in B_1$ and $b_2 \in B_2$. If $b_1 \otimes b_2$ is a highest (respectively, lowest) weight vector in $B_1 \otimes B_2$, then b_1 is a highest weight vector (respectively, b_2 is a lowest weight vector).
(iii) Let $b \in B(\lambda)$. In the tensor product $B(\mu) \otimes B(\lambda)$, the element $\bar{u}_\mu \otimes b$ is a highest weight vector if and only if $\mu - \varepsilon(b)$ is a dominant weight. If this condition holds, then the connected component of $B(\mu) \otimes B(\lambda)$ to which $\bar{u}_\mu \otimes b$ belongs is isomorphic to $B(\mu + \mathrm{wt}(b))$.
(iv) In the tensor product $B(\lambda) \otimes B(\mu)$, the elements $\bar{u}_\lambda \otimes \bar{u}_\mu$ and $\bar{d}_\lambda \otimes \bar{d}_\mu$ are the highest and the lowest weight vectors of the same connected component.
(v) In the tensor product $B(\lambda) \otimes B(\mu)$, there is a connected component isomorphic to $B(0)$ if and only if $\lambda = -w_0\mu$. If this condition holds, then that component is reduced to the element $\bar{u}_\lambda \otimes \bar{d}_\mu$.
(vi) Given two normal crystals B_1 and B_2, with B_1 connected, a morphism g from B_1 to B_2 is either the zero map or an isomorphism onto a connected component of B_2.

Proof. The construction of the crystal basis of $V(\lambda)$ shows that any element b in $B(\lambda)$ can be written as $\tilde{f}_{i_1} \cdots \tilde{f}_{i_n}(\bar{u}_\lambda)$ for some finite sequence $(i_1, \ldots, i_n) \in I$. If $b \neq \bar{u}_\lambda$, then $n \geq 1$ and so $\tilde{e}_{i_1}(b) \neq 0$, which shows that b is not a highest weight vector. Thus $\bar{u}_\lambda$ is the only highest weight vector in $B(\lambda)$. Consider the crystal $B(\lambda)^\vee$ obtained from $B(\lambda)$ by exchanging the action of the operators $\tilde{e}_i$ with that of the corresponding operators $\tilde{f}_i$. The argument given in Section 7.4 of [Ka3] proves the existence of an isomorphism from the crystal $B(-w_0\lambda)$ onto $B(\lambda)^\vee$ which sends $\bar{u}_{-w_0\lambda}$ to $\bar{d}_\lambda$. Therefore $\bar{d}_\lambda$ is the only lowest weight vector in $B(\lambda)$. Statement (i) is proved.

Statement (ii) and the first assertion in Statement (iii) are direct consequences of the rules (5) and (6) that define the maps $\tilde{e}_i$ and $\tilde{f}_i$ on a tensor product of two crystals. Now adopt the notation of Statement (iii), and suppose that $\bar{u}_\mu \otimes b$ is a highest weight vector. The connected component of $B(\mu) \otimes B(\lambda)$ to which $\bar{u}_\mu \otimes b$ belongs is isomorphic to a crystal $B(\nu)$, for some $\nu \in P_{++}$. The isomorphism sends $\bar{u}_\mu \otimes b$ to $\bar{u}_\nu$, and so $\nu = \mathrm{wt}(\bar{u}_\mu \otimes b) = \mu + \mathrm{wt}(b)$. This proves the second assertion in Statement (iii).

The simple module $V(\lambda+\mu)$ has multiplicity one in $V(\lambda)\otimes V(\mu)$ therefore there is one connected component isomorphic to $B(\lambda + \mu)$ in $B(\lambda) \otimes B(\mu)$, by Proposition 1 (iv). Since $\bar{u}_\lambda \otimes \bar{u}_\mu$ and $\bar{d}_\lambda \otimes \bar{d}_\mu$ are the only elements of $B(\lambda) \otimes B(\mu)$ with weights $\lambda + \mu$ and $w_0(\lambda + \mu)$, respectively, they belong to this component. Statement (iv) follows.

The trivial $U_q(\mathfrak{g})$-module $V(0)$ arises as a submodule of $V(\lambda)\otimes V(\mu)$ if and only if $\lambda = -w_0\mu$. Under this assumption, there is one connected component isomorphic to $B(0)$ in $B(\lambda) \otimes B(\mu)$. This component is reduced to a single element, which is a highest and a lowest weight vector. By Statement (ii), this element is $\bar{u}_\lambda \otimes \bar{d}_\mu$. This proves Statement (v).

Let finally g be a non-zero morphism from a connected normal crystal B_1 to a normal crystal B_2. The crystal $g(B_1) \setminus \{0\}$ is connected, hence is contained in a connected component of B_2. On the other hand, the operators $\tilde{e}_i$ and $\tilde{f}_i$ send $g(B_1) \cup \{0\}$ into itself, so $g(B_1)$ contains every connected component of B_2 that it meets. We conclude that $g(B_1) \setminus \{0\}$ is a connected component B' of B_2. By Statement (i), the crystal B' has a unique highest weight vector, which is the image through g of the highest weight vector of B_1. In particular these highest weight vectors have the same weight, which implies that B_1 and B' are isomorphic to the same $B(\lambda)$, by Proposition 1 (iv). Therefore B_1 and B' have the same finite cardinality, and g is a bijection from B_1 onto B'. Statement (vi) is proved. □

3. Proof of Theorem 2

3.1. Sketch of the proof

Let us explain the construction of $\mathcal{H}_q$. We want the multiplication to define an isomorphism of $U_q(\mathfrak{g})$-modules from $\mathbb{C}_q[G]^G \otimes \mathcal{H}_q$ onto $\mathbb{C}_q[G]$. Therefore we must investigate the structure of the $U_q(\mathfrak{g})$-module $\mathbb{C}_q[G]$.

The multiplicities of the simple modules inside the tensor product $V(\mu) \otimes V(\lambda)$ are given by the generalized Littlewood-Richardson rule (Proposition 4.2 in [Ka3] or Theorem 6.4.16 in [Jo]; see also [Li]):

$$V(\mu) \otimes V(\lambda) \simeq \bigoplus_{\substack{b \in B(\lambda) \\ \mu - \varepsilon(b) \in P_{++}}} V(\mu + \mathrm{wt}(b)). \tag{7}$$

Since the module $V(\lambda)$ is isomorphic to its bidual $V(\lambda)^{**}$, we have

$$\begin{aligned} \mathrm{Hom}_{U_q(\mathfrak{g})}(V(\lambda)^*, C(\mu)) &\simeq (V(\mu)^* \otimes V(\mu) \otimes V(\lambda))^{U_q(\mathfrak{g})} \\ &\simeq \bigoplus_{\substack{b \in B(\lambda) \\ \mu - \varepsilon(b) \in P_{++}}} (V(\mu)^* \otimes V(\mu + \mathrm{wt}(b)))^{U_q(\mathfrak{g})}, \end{aligned}$$

and thus there is a basis $(r_b(\mu))$ of $\mathrm{Hom}_{U_q(\mathfrak{g})}(V(\lambda)^*, C(\mu))$ indexed by the set $\{b \in B(\lambda)_0 \mid \mu - \varepsilon(b) \in P_{++}\}$. We denote the image of $r_b(\mu)$ by $F_b(\mu)$. Then the module $C(\mu)$ is the direct sum of the submodules $F_b(\mu)$, for those $b \in \bigsqcup_{\lambda \in P_{++}} B(\lambda)_0$ such that $\mu - \varepsilon(b) \in P_{++}$.

In each subspace $C(\nu)$, there is a unique (up to a scalar) $U_q(\mathfrak{g})$-invariant vector, the so-called quantum trace in the module $V(\lambda)$, which we denote by ${\mathrm{Tr}_q}^\nu$. (We do not need more information about this element; for completeness however, we recall that this quantum trace is the linear form on $U_q(\mathfrak{g})$ whose value on an element x is the trace of the operator defined by the element $xK_{2\rho} \in U_q(\mathfrak{g})$ on the module $V(\nu)$, where $\rho = \sum_{i \in I} \varpi_i$, see for instance Lemma 7.1.18 in [Jo].) If for any $b \in B(\lambda)_0$ and any $\nu \in P_{++}$, the multiplication by ${\mathrm{Tr}_q}^\nu$ sent the subspace $F_b(\varepsilon(b))$ of $C(\varepsilon(b))$ onto the subspace $F_b(\varepsilon(b)+\nu)$ of $C(\varepsilon(b)+\nu)$, then, letting $\mathcal{H}_q$ be the sum of the spaces $F_b(\varepsilon(b))$, where b runs over $\bigsqcup_{\lambda\in P_{++}} B(\lambda)_0$, we would have

$$\begin{aligned}
\mathbb{C}_q[G] &= \bigoplus_{\mu\in P_{++}} \Bigg(\bigoplus_{\substack{b\in\bigsqcup_\lambda B(\lambda)_0 \\ \text{such that } \mu-\varepsilon(b)\in P_{++}}} F_b(\mu) \Bigg) \\
&= \bigoplus_{b\in\bigsqcup_\lambda B(\lambda)_0} \bigoplus_{\nu\in P_{++}} F_b(\varepsilon(b)+\nu) \\
&= \Bigg(\bigoplus_{\nu\in P_{++}} {\mathrm{Tr}_q}^\nu \Bigg) \Bigg(\bigoplus_{b\in\bigsqcup_\lambda B(\lambda)_0} F_b(\varepsilon(b)) \Bigg) \\
&= \mathbb{C}_q[G]^G\, \mathcal{H}_q,
\end{aligned}$$

as required.

However this does not work so easily. First, Equation (7) gives only the multiplicities in the tensor product, and the proper definition of the submodules $F_b(\mu)$ requires some additional work; this problem will be handled in Sections 3.3 and 3.4. Second, the multiplication by ${\mathrm{Tr}_q}^\nu$ does not send $C(\mu)$ into $C(\mu+\nu)$. The trouble will be cured with the help of a filtration, which we will define in Section 3.2. (A similar filtration was used in the original proof of Theorem 1 by Joseph and Letzter.) Third, even the use of this filtration does not ensure that the multiplication by ${\mathrm{Tr}_q}^\nu$ sends $F_b(\varepsilon(b))$ onto $F_b(\varepsilon(b)+\nu)$. Indeed this latter fact is true only at the level of crystals, as will be shown in Section 3.4.

To sum up, the next sections give the precise definitions needed for a complete proof.

3.2. A filtration on $\mathbb{C}_q[G]$

We define an order relation on the semigroup P_{++} by saying that $\lambda \geq \mu$ whenever $\lambda-\mu\in Q_+$. By its definition (see Equation (1)), the coadjoint $U_q(\mathfrak{g})$-module $\mathbb{C}_q[G]$ is graded by the semigroup P_{++}. It is well-known that the corresponding filtration on $\mathbb{C}_q[G]$:

$$\mathbb{C}_q[G]_\lambda = \bigoplus_{\mu\leq\lambda} C(\mu)$$

is a filtration of algebras. The associated graded algebra, denoted by $\mathrm{gr}(\mathbb{C}_q[G])$, is, as a $U_q(\mathfrak{g})$-module, canonically isomorphic to $\mathbb{C}_q[G]$.

For any $\mu\in P_{++}$, we chose a highest weight vector u_μ in the module $V(\mu)$. For any $\mu,\nu\in P_{++}$, let us denote the $U_q(\mathfrak{g})$-linear map from $V(\mu)\otimes V(\nu)$ to $V(\mu+\nu)$ that sends $u_\mu\otimes u_\nu$ to $u_{\mu+\nu}$ by $p_{\mu,\nu}$. Set

$$E^\mu = V(-w_0\mu)\otimes V(\mu) \quad \text{for } \mu\in P_{++}, \quad \text{and} \quad E=\bigoplus_{\mu\in P_{++}} E^\mu.$$

The family of linear maps

$$\begin{pmatrix} E^\mu \otimes E^\nu & \to & E^{\mu+\nu} \\ (m\otimes n)\otimes(p\otimes q) & \mapsto & p_{-w_0\nu,-w_0\mu}(p\otimes m)\otimes p_{\mu,\nu}(n\otimes q) \end{pmatrix} \tag{8}$$

endows E with the structure of an algebra (a priori non-associative and without unit).

Now for each weight $\mu \in P_{++}$, there is an isomorphism of $U_q(\mathfrak{g})$-modules

$$g_\mu : V(-w_0\mu) \to V(\mu)^* \quad \text{such that} \quad \langle g_\mu(u_{-w_0\mu}), d_\mu\rangle = 1.$$

It gives rise to an isomorphism of $U_q(\mathfrak{g})$-modules

$$h_\mu : \left(V(-w_0\mu)\otimes V(\mu) \to C(\mu),\ \ell\otimes m \mapsto c^{V(\mu)}_{g_\mu(\ell),m}\right).$$

Lemma 2. *There exists a family of scalars $(\zeta_\mu) \in \left(\mathbb{C}(q)^\times\right)^{P_{++}}$ such that the map $\bigoplus_{\mu\in P_{++}} \zeta_\mu h_\mu : E \to \mathrm{gr}(\mathbb{C}_q[G])$ is a $U_q(\mathfrak{g})$-linear isomorphism of algebras.*

Proof. For $\mu,\nu \in P_{++}$, we denote by $i_{\mu,\nu}$ the $U_q(\mathfrak{g})$-linear map from $V(\mu+\nu)$ to $V(\mu)\otimes V(\nu)$ that sends $d_{\mu+\nu}$ to $d_\mu \otimes d_\nu$. We denote its dual map by $(i_{\mu,\nu})^T : V(\nu)^*\otimes V(\mu)^* \to V(\mu+\nu)^*$. The following identity can be easily checked:

$$g_{\mu+\nu}\circ p_{-w_0\nu,-w_0\mu} = (i_{\mu,\nu})^T \circ (g_\nu\otimes g_\mu).$$

Being a non-zero map of the absolutely simple $U_q(\mathfrak{g})$-module $V(\mu+\nu)$, the map $p_{\mu,\nu}\circ i_{\mu,\nu}$ is a scalar automorphism $\tau_{\mu,\nu}\,\mathrm{id}_{V(\mu+\nu)}$. The equalities

$$p_{\mu+\nu,\sigma}\circ(p_{\mu,\nu}\otimes \mathrm{id}_{V(\sigma)}) = p_{\mu,\nu+\sigma}\circ(\mathrm{id}_{V(\mu)}\otimes p_{\nu,\sigma}),$$
$$(i_{\mu,\nu}\otimes \mathrm{id}_{V(\sigma)})\circ i_{\mu+\nu,\sigma} = (\mathrm{id}_{V(\mu)}\otimes i_{\nu,\sigma})\circ i_{\mu,\nu+\sigma}$$

lead to $\tau_{\mu+\nu,\sigma}\tau_{\mu,\nu} = \tau_{\mu,\nu+\sigma}\tau_{\nu,\sigma}$, for all $\mu,\nu,\sigma\in P_{++}$. Therefore there exist elements $\zeta_\mu \in \mathbb{C}(q)^\times$ such that $\zeta_\mu\zeta_\nu = \tau_{\mu,\nu}\zeta_{\mu+\nu}$, for all μ and $\nu\in P_{++}$, because P_{++} is a free abelian semigroup.

Now for any $(m\otimes n)\in E^\mu$ and $(p\otimes q)\in E^\nu$ as in the definition of the multiplication map (8), we compute:

$$\begin{aligned}
h_{\mu+\nu}\left((m\otimes n)(p\otimes q)\right) &= c^{V(\mu+\nu)}_{(g_{\mu+\nu}\circ p_{-w_0\nu,-w_0\mu})(p\otimes m),p_{\mu,\nu}(n\otimes q)} \\
&= c^{V(\mu+\nu)}_{(i_{\mu,\nu})^T\circ(g_\nu\otimes g_\mu)(p\otimes m),p_{\mu,\nu}(n\otimes q)} \\
&= c^{V(\mu)\otimes V(\nu)}_{g_\nu(p)\otimes g_\mu(m),(i_{\mu,\nu}\circ p_{\mu,\nu})(n\otimes q)} \\
&\equiv \tau_{\mu,\nu}\, c^{V(\mu)}_{g_\mu(m),n}\, c^{V(\nu)}_{g_\nu(p),q} \qquad \mathrm{mod} \bigoplus_{\sigma<\mu+\nu} C(\sigma) \\
&\equiv \tau_{\mu,\nu}\, h_\mu(m\otimes n)\, h_\nu(p\otimes q) \qquad \mathrm{mod} \bigoplus_{\sigma<\mu+\nu} C(\sigma).
\end{aligned}$$

A comparison with the relations $\tau_{\mu,\nu} = \zeta_\mu \zeta_\nu / \zeta_{\mu+\nu}$ completes the proof. The ζ_μ are necessary to rectify our careless choice of the vectors d_μ in Section 2.6. □

The following statement, which clarifies somewhat the properties of the multiplication map of E, is a direct consequence of Lemmas 1 and 2.

Lemma 3. *The algebra E is associative and has a unit. For any invariant element c in the $U_q(\mathfrak{g})$-module E, the map $(E \to E,\ x \mapsto cx)$ is $U_q(\mathfrak{g})$-linear.*

3.3. Indexation of the connected components of $B(-w_0\mu) \otimes B(\mu)$

Let us choose $\lambda, \mu \in P_{++}$ and $b \in B(\lambda)_0$ such that $\mu - \varepsilon(b) \in P_{++}$. By Proposition 2 (iii), the element $\bar{u}_\mu \otimes b$ is a highest weight vector of weight μ in the crystal $B(\mu) \otimes B(\lambda)$. The connected component to which it belongs is isomorphic to $B(\mu)$ and contains a unique lowest weight vector, which is, by Proposition 2 (ii), of the form $t_b(\mu) \otimes \bar{d}_\lambda$, where $t_b(\mu) \in B(\mu)_{w_0(\mu-\lambda)}$. Then Proposition 2 (v) shows that in the crystal $B(-w_0\mu) \otimes B(\mu) \otimes B(\lambda)$, the element $\bar{u}_{-w_0\mu} \otimes t_b(\mu) \otimes \bar{d}_\lambda$ spans its connected component, which is isomorphic to $B(0)$. Thus, again by Proposition 2 (v), the element $\bar{u}_{-w_0\mu} \otimes t_b(\mu)$ is a highest weight vector of weight $-w_0\lambda$ in the crystal $B(-w_0\mu) \otimes B(\mu)$. We denote the connected component to which it belongs by $W_b(\mu)$; the crystal $W_b(\mu)$ is isomorphic to $B(-w_0\lambda)$.

Lemma 4. *(i) Let $\mu \in P_{++}$. The assignment $b \mapsto W_b(\mu)$ is a bijection from the set $\{b \in \bigsqcup_{\lambda \in P_{++}} B(\lambda)_0 \mid \mu - \varepsilon(b) \in P_{++}\}$ onto the set of connected components of the crystal $B(-w_0\mu) \otimes B(\mu)$.*
(ii) For any $\lambda, \mu, \nu \in P_{++}$ and $b \in B(\lambda)_0$ such that $\mu - \varepsilon(b) \in P_{++}$, the morphism of crystals $\bar{p}_{\nu,\mu} : B(\nu) \otimes B(\mu) \to B(\mu + \nu)$, induced by the map $p_{\nu,\mu}$, sends $\bar{d}_\nu \otimes t_b(\mu)$ to $t_b(\mu + \nu)$.

Proof. Using Proposition 2, one can easily see that all the steps used in the definition of the map $b \mapsto W_b(\mu)$ are reversible. Assertion (i) follows.

Let us turn to Assertion (ii). The elements $\bar{u}_\mu \otimes b$ and $t_b(\mu) \otimes \bar{d}_\lambda$ are the highest and the lowest weight vectors of the same connected component of the crystal $B(\mu) \otimes B(\lambda)$. Therefore by Proposition 2 (iv), the elements $\bar{u}_\nu \otimes \bar{u}_\mu \otimes b$ and $\bar{d}_\nu \otimes t_b(\mu) \otimes \bar{d}_\lambda$ are the highest and the lowest weight vectors of the same connected component of the crystal $B(\nu) \otimes B(\mu) \otimes B(\lambda)$. By Proposition 1 (iii), the $U_q(\mathfrak{g})$-linear map $p_{\mu,\nu} \otimes \mathrm{id}_{V(\lambda)}$ induces a morphism of crystals $\bar{p}_{\nu,\mu} \otimes \mathrm{id}_{B(\lambda)}$ from $B(\nu) \otimes B(\mu) \otimes B(\lambda)$ to $B(\mu+\nu) \otimes B(\lambda)$, which sends that connected component isomorphically onto a connected component of $B(\mu + \nu) \otimes B(\lambda)$, by Proposition 2 (vi). The highest weight vector of this latter is then $\bar{p}_{\nu,\mu}(\bar{u}_\nu \otimes \bar{u}_\mu) \otimes b = \bar{u}_{\mu+\nu} \otimes b$ and the lowest weight one is $\bar{p}_{\nu,\mu}(\bar{d}_\nu \otimes t_b(\mu)) \otimes \bar{d}_\lambda$. The result follows. □

3.4. Completion of the proof

We endow each $U_q(\mathfrak{g})$-module E^μ with the crystal basis

$$(\mathcal{L}^\mu, \mathcal{B}^\mu) = (L(-w_0\mu) \otimes_A L(\mu), B(-w_0\mu) \otimes B(\mu)),$$

and get a crystal basis $(\mathcal{L}, \mathcal{B})$ of E by forming their direct sum. By Proposition 1 (ii), the pair $(\mathcal{L}, \mathcal{B})$ defines a crystal basis $(\mathcal{L}', \mathcal{B}')$ of the subspace $E^{U_q(\mathfrak{g})}$ of invariant vectors in the module E.

Lemma 4 (i) says that for each $\mu \in P_{++}$, the crystal $B(-w_0\mu) \otimes B(\mu)$ is the direct sum of the subcrystals $W_b(\mu)$, where b runs over $\{b \in \bigsqcup_\lambda B(\lambda)_0 \mid \mu - \varepsilon(b) \in P_{++}\}$. By Proposition 1 (i), we can therefore find, for each $\mu \in P_{++}$, a decomposition of the module E^μ as a direct sum of simple submodules $F_b(\mu)$ such that

- $(F_b(\mu) \cap \mathcal{L}^\mu, W_b(\mu))$ is a crystal basis of $F_b(\mu)$;
- $\mathcal{L}^\mu = \bigoplus_b (F_b(\mu) \cap \mathcal{L}^\mu)$;

where again b runs over $\{b \in \bigsqcup_\lambda B(\lambda)_0 \mid \mu - \varepsilon(b) \in P_{++}\}$. We define F to be the direct sum of the submodules $F_b(\varepsilon(b))$ of E, for all $b \in \bigsqcup_\lambda B(\lambda)_0$, and endow it with the crystal basis

$$(\mathcal{L}_F, \mathcal{B}_F) = \bigoplus_b \Big(F_b(\varepsilon(b)) \cap \mathcal{L}^{\varepsilon(b)}, W_b(\varepsilon(b))\Big).$$

Lemma 5. *The multiplication in E affords an isomorphism of $U_q(\mathfrak{g})$-modules from $E^{U_q(\mathfrak{g})} \otimes_{\mathbb{C}(q)} F$ onto E, which carries the crystal basis*

$$(\mathcal{L}' \otimes_A \mathcal{L}_F, \mathcal{B}' \otimes \mathcal{B}_F) \quad \textit{onto} \quad (\mathcal{L}, \mathcal{B}).$$

Proof. Let f be the restriction to $E^{U_q(\mathfrak{g})} \otimes_{\mathbb{C}(q)} F$ of the multiplication map of E. Lemma 3 implies that f is $U_q(\mathfrak{g})$-linear, because the $U_q(\mathfrak{g})$-module $E^{U_q(\mathfrak{g})}$ is trivial. The definition (8) of the multiplication on E and Proposition 1 (iii) imply that f is compatible with the crystal bases considered above. By Proposition 2 (v), $\mathcal{B}'$ is the subcrystal consisting of the elements $\bar{u}_{-w_0\nu} \otimes \bar{d}_\nu$ in $\mathcal{B} = \bigoplus_\nu (B(-w_0\nu) \otimes B(\nu))$.

Let us take a connected component $W_b(\varepsilon(b))$ of $\mathcal{B}_F$, where $b \in B(\lambda)_0$ for some $\lambda \in P_{++}$. Since the crystal $B(0) \otimes B(-w_0\lambda)$ is isomorphic to $B(-w_0\lambda)$, hence connected, the crystal $\{\bar{u}_{-w_0\nu} \otimes \bar{d}_\nu\} \otimes W_b(\varepsilon(b))$ is connected and its highest weight vector is $\bar{u}_{-w_0\nu} \otimes \bar{d}_\nu \otimes \bar{u}_{-w_0\varepsilon(b)} \otimes t_b(\varepsilon(b))$. By Lemma 4 (ii), the morphism $\bar{f}$ sends this element to

$$\bar{p}_{-w_0\varepsilon(b),-w_0\nu}(\bar{u}_{-w_0\varepsilon(b)} \otimes \bar{u}_{-w_0\nu}) \otimes \bar{p}_{\nu,\varepsilon(b)}(\bar{d}_\nu \otimes t_b(\varepsilon(b))) \\ = \bar{u}_{-w_0(\varepsilon(b)+\nu)} \otimes t_b(\varepsilon(b)+\nu),$$

hence to the highest weight vector of the connected component $W_b(\varepsilon(b)+\nu)$ of $\mathcal{B}$. According to Proposition 2 (vi), this means that the morphism $\bar{f}$ sends isomorphically $\{\bar{u}_{-w_0\nu} \otimes \bar{d}_\nu\} \otimes W_b(\varepsilon(b))$ onto $W_b(\varepsilon(b)+\nu)$. Therefore we have a commutative diagram

$$\begin{tikzcd}
\mathcal{L}' \otimes_A \mathcal{L}_F \arrow[r, "f"] \arrow[d] & \mathcal{L} \arrow[d] \\
(\mathcal{L}' \otimes_A \mathcal{L}_F)/q(\mathcal{L}' \otimes_A \mathcal{L}_F) \arrow[r, "\bar{f}"] & \mathcal{L}/q\mathcal{L}
\end{tikzcd}$$

where the vertical maps are the canonical surjections and where, by Lemma 4 (i), the bottom line is an isomorphism of $\mathbb{C}$-vector spaces. Now $\mathcal{L}' \otimes_A \mathcal{L}_F$ and $\mathcal{L}$ are P_{++}-graded free A-modules, whose graded components are of finite rank, and f preserves the grading. We conclude that the top line is an isomorphism of A-modules, and therefore that f is an isomorphism of $U_q(\mathfrak{g})$-modules. $\square$

Let us call $\widetilde{\mathcal{H}}_q \subseteq \mathrm{gr}(\mathbb{C}_q[G])$ the image of F through the isomorphism $\bigoplus \zeta_\mu h_\mu$ obtained in Lemma 2, and lift $\widetilde{\mathcal{H}}_q$ to a submodule $\mathcal{H}_q$ of $\mathbb{C}_q[G]$ using the canonical isomorphism of $U_q(\mathfrak{g})$-modules $\mathbb{C}_q[G] \xrightarrow{\sim} \mathrm{gr}(\mathbb{C}_q[G])$. The multiplication in $\mathbb{C}_q[G]$ defines a map

$$\mathbb{C}_q[G]^G \otimes_{\mathbb{C}(q)} \mathcal{H}_q \to \mathbb{C}_q[G]$$

which is $U_q(\mathfrak{g})$-linear by Lemma 1, and bijective, since its graded counterpart is bijective by Lemmas 2 and 5.

Finally we remark that the multiplicity of the simple module $V(-w_0\lambda)$ in $\mathcal{H}_q$ is the same as in $\widetilde{\mathcal{H}}_q$ or in F, hence is equal to the cardinality of $B(\lambda)_0$, i.e. to the dimension of the zero-weight space $V(-w_0\lambda)_0$. This concludes the proof of Theorem 2.

3.5. Final comments

The module $\mathcal{H}_q$ constructed in Section 3.4 is not uniquely determined, since the decompositions $E^\mu = \bigoplus_b F_b(\mu)$ are only specified at the crystal limit. However the global crystal bases of the spaces E^μ defined in Section 2.1 of [Ka2] can be used to gain uniqueness. (In the actual procedure, one has to project the dual bases of these global crystal bases onto the isotypical components of E^μ.) It should then be possible to show that the isomorphism from $\mathbb{C}_q[G]^G \otimes_{\mathbb{C}(q)} \mathcal{H}_q$ onto $\mathbb{C}_q[G]$ can be specialized at $q = 1$, which would yield another proof of Richardson's theorem.

The "Killing isomorphism" I becomes degenerate at the classical limit $q = 1$, and it seems difficult to deduce Kostant's theorem from Richardson's one by a method similar to the one presented in Section 2.4.

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