COBORDANT ALGEBRAIC KNOTS DEFINED BY BRIESKORN POLYNOMIALS

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ABSTRACT. In this paper we go further in the study of cobordism of algebraic knots associated with Brieskorn polynomials initiated in [3]. We define new sets of invariants for these cobordism classes. Using these invariants we find more examples of distinct cobordism classes with distinct exponents.

1. INTRODUCTION

A *Brieskorn polynomial* is a polynomial of the form

$$P(\mathbf{z}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$

with $\mathbf{z} = (z_1, z_2, \ldots, z_{n+1})$, $n \geq 1$, where the integers $a_j \geq 2$, $j = 1, 2, \ldots, n+1$, are called the *exponents*. The complex hypersurface in $\mathbf{C}^{n+1}$ defined by $P = 0$ has an isolated singularity at the origin, which is called a *Brieskorn singularity*.

To be more precise, let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a holomorphic function germ with an isolated critical point at the origin. We denote by D_ε^{2n+2} the closed ball of radius $\varepsilon > 0$ centred at 0 in $\mathbf{C}^{n+1}$, and by S_ε^{2n+1} its boundary. According to Milnor [12], the oriented homeomorphism class of the pair $(D_\varepsilon^{2n+2}, f^{-1}(0) \cap D_\varepsilon^{2n+2})$ does not depend on the choice of a sufficiently small $\varepsilon > 0$, and by definition it is the *topological type* of f. The oriented diffeomorphism class of the pair $(S_\varepsilon^{2n+1}, K_f)$, with $K_f = f^{-1}(0) \cap S_\varepsilon^{2n+1}$, is the *algebraic knot* associated with f, where K_f is a closed oriented $(2n-1)$-dimensional manifold. According to Milnor's cone structure theorem [12], the algebraic knot K_f determines the topological type of f. In fact, it is known that the converse also holds.

Definition 1.1. An *m-dimensional knot*, or a *m-knot*, is a closed oriented m-dimensional submanifold of the oriented $(m+2)$-dimensional sphere S^{m+2}. When this submanifold is homeomorphic to a sphere we call this a *sperical knot*. Two m-knots K_0 and K_1 in S^{m+2} are said to be *cobordant* if there exists a properly embedded oriented $(m+1)$-dimensional submanifold X of $S^{m+2} \times [0, 1]$ such that

(1) X is diffeomorphic to $K_0 \times [0, 1]$, and
(2) $\partial X = (K_0 \times \{0\}) \cup (-K_1 \times \{1\})$,

where $-K_1 \times \{1\}$ denotes the manifold $K_1 \times \{1\}$ with the reversed orientation. A manifold X as above is called a *cobordism* between K_0 and K_1 (see Fig. 1).

In this paper, we will study the cobordism classes of algebraic knots associated with Brieskorn singularities, we will call such knots *Brieskorn knots* for short.

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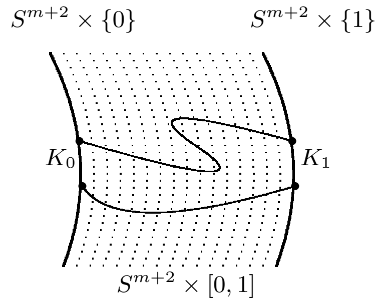

FIGURE 1. A cobordism between K_0 and K_1

Moreover, our goal is to study these cobordism classes in terms of exponents of the Brieskorn polynomials. In [3] we proved that when n equals 1 or 2 Brieskorn knots are cobordant if and only if they have same exponents. Then, we will work with $n \geq 3$.

In [1], for $n \geq 3$, necessary and sufficient conditions for two algebraic $(2n-1)$-knots to be cobordant have been obtained in terms of Seifert forms (for the definition of the Seifert form, see §2). However, the computation of the Seifert form of a given algebraic knot is very difficult, and an explicit calculation of a Seifert form is known only for a very limited class of algebraic knots. Furthermore, even if we know the Seifert forms explicitly, it is still difficult to see if given two such forms satisfy the algebraic conditions given in [1] or not.

When the knots are spherical, the algebraic condition of cobordism becomes much more simple. M. Kervaire [9] and J. Levine [10] proved that spherical knot are cobordant if and only if they have Witt equivalent Seifert forms and J. Levine [11] gave a complete list of invariants for cobordism classes of spherical knots. (for details, see §2).

Recall that cobordism does not necessarily imply isotopy for algebraic knots in general. For details, see the survey article [2].

In this paper, we first associate some spherical Brieskorn knots to a given Brieskorn knot. Since the set of exponents of such associated spherical knots are very similar to which of the Brieskorn knot, the study of cobordism classes of these spherical Brieskorn knots impose conditions on the exponents' sets of the initial Brieskorn cobordism class.

The paper is organized as follows. In §2 we give some definitions and classical results, then in §3 we give a new set of invariants for cobordism classes of Brieskorn knots.

Throughout the paper we work in the smooth category. All the homology groups are with integer coefficients unless otherwise specified.

2. Definitions and results on spherical knots cobordism

Let $f(\mathbf{z})$ be a polynomial in $\mathbf{C}^{n+1}$ with an isolated critical point at the origin. We denote by F_f the *Milnor fiber* associated with f, i.e., F_f is the closure of a fiber of the Milnor fibration $\varphi_f : S_\varepsilon^{2n+1} \setminus K_f \to S^1$ defined by $\varphi_f(\mathbf{z}) = f(\mathbf{z})/|f(\mathbf{z})|$. According to Milnor [12], F_f is a compact $2n$-dimensional submanifold of S_ε^{2n+1}

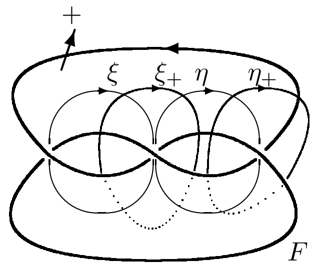

FIGURE 2. Computing a Seifert matrix for the trefoil knot

which is homotopy equivalent to the bouquet of a finite number of copies of the n-dimensional sphere.

The Seifert form

$$L_f : H_n(F_f) \times H_n(F_f) \to \mathbf{Z}$$

associated with f is defined by

$$L_f(\alpha, \beta) = \mathrm{lk}(a_+, b),$$

where a and b are n-cycles representing α and β in $H_n(F_f)$ respectively, a_+ is the n-cycle in S^{2n+1}_ε obtained by pushing a into the positive normal direction of F_f, and lk denotes the linking number of n-cycles in S^{2n+1}_ε. See Fig. 2 for a picture of cycles necessary to compute the trefoil knot's Seifert from. It is known that the isomorphism class of the Seifert form is a topological invariant of f. Furthermore, two algebraic knots K_f and K_g associated with polynomials f and g in $\mathbf{C}^{n+1}$, respectively, with isolated critical points at the origin are isotopic in S^{2n+1}_ε if and only if their Seifert forms L_f and L_g are isomorphic, provided that $n \geq 3$.

In fact, algebraic knots are *simple fibered* knots as follows. We say that an oriented m-knot K is *fibered* if there exists a smooth fibration $\phi : S^{m+2} \setminus K \to S^1$ and a trivialization $\tau : N_K \to K \times D^2$ of a closed tubular neighborhood N_K of K in S^{m+2} such that $\phi|_{N_K \setminus K}$ coincides with $\pi \circ \tau|_{N_K \setminus K}$, where $\pi : K \times (D^2 \setminus \{0\}) \to S^1$ is the composition of the projection to the second factor and the obvious projection $D^2 \setminus \{0\} \to S^1$. Note that then the closure of each fiber of ϕ in S^{m+2} is a compact $(m+1)$-dimensional oriented manifold whose boundary coincides with K. We shall often call the closure of each fiber simply a *fiber*. Moreover, for $m = 2n-1 \geq 1$ we say that a fibered $(2n-1)$-knot K is *simple* if each fiber of ϕ is $(n-1)$-connected and K is $(n-2)$-connected. For details we refer the reader to [2]. Note that two simple fibered $(2n-1)$-knots are isotopic if and only if they have isomorphic Seifert forms, provided $n \geq 3$ (see [7, 8]).

Definition 2.1. Two bilinear forms $L_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, defined on free abelian groups G_i of finite ranks are said to be *Witt equivalent* if there exists a direct summand M of $G_0 \oplus G_1$ such that $(L_0 \oplus (-L_1))(x, y) = 0$ for all $x, y \in M$ and twice the rank of M is equal to the rank of $G_0 \oplus G_1$. In this case, M is called a *metabolizer*.

2.1. **Characterization of spherical Brieskorn knots.** We refer to [4] for the results of this subsection, the reader may consult the book [6] for detailed proofs as well.

Let $P(\mathbf{z}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$ be a Brieskorn polynomial, we associate to this polynomial a graph G_P. This graph has $n+1$ vertices labeled by the letters $a_1, \ldots, a_{n+1}$, and two vertices a_i and a_j are connected with and edge if and only if $\gcd(a_i, a_j)$ is strictly greater to 1. We denote by $C_{ev,P}$ the connected component of G_P wich contain all odd exponents ; remark that $C_{ev,P}$ may contain some odd labeled vertices as well. We say that $C_{ev,P}$ fulfills condition $\mathscr{C}$ if it contains an odd number of vertices and $\gcd(a_i, a_j) = 2$ for any two distinct points a_i and a_j in $C_{ev,P}$.

Theorem 2.2 ([4])**.** *Let $n \geq 3$. The Brieskorn knot associated to P is spherical if and only if the graph G_P contains at least two isolated points or an isolated point which is odd and the component $C_{ev,P}$ fulfills condition $\mathscr{C}$.*

Let K_P be the Brieskorn knot associated to the Brieskorn polynomial

$$P(\mathbf{z}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}.$$

Set p and q be two distinct prime numbers such that for all $i = 1, \ldots, n+1$ we have $\gcd(p, a_i) = \gcd(q, a_i) = \gcd(p, q) = 1$. Then we define the following polynomial

$$P_{p,q}(\mathbf{z}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}} + z_{n+2}^{p} + z_{n+3}^{q}.$$

According to Theorem 2.2 and h-cobordism theorem [16], the Brieskorn knot $K_{P_{p,q}}$ is spherical. Note that K_P and $K_{P_{p,q}}$ are respectively of dimension $2n+1$ and $2n+3$.

2.2. **Levine's complete invariant set of cobordant isometric structures.** We refer to [11] for all results of this subsection.

In order to study C_n the cobordism group ok knotted n-dimensional spheres in codimension two, J. Levine introduced *isometric structures* $\big(<,>, T\big)$ where $<,>$ is a non degenerate symmetric bilinear form on a finite dimensional vector space V over $\mathbf{Q}$, and T is an isometry of V. Recall that the group law of C_n is the connected sum ; two such knots K_0 and K_1 are cobordant if the oriented connected sum $K_0 \# -K_1$ is null-cobordant, i.e. cobordant to the trivial embedding, and this only happens when the associated Seifert forms are Witt-equivalent.

Following [11], an isometric structure $\big(<,>, T\big)$ is null-cobordant when V contains a totally isotropic subspace, invariant under T, of half dimension of V. Moreover, two isometric structures $\big(<,>, T\big)$ and $\big(<,>', T'\big)$ are *cobordant* if the orthogonal sum $\big(<,>, T\big) \perp \big(- <,>', T'\big)$ is null-cobordant. As in the case of spherical knots, this equivalence relation gives an abelian group of cobordism classes of isometric structures.

Let $\Delta_T(t)$ be the characteristic polynomial of T. Levine proved that the group of cobordism classes of isometric structures $\big(<,>, T\big)$ satisfying $\Delta_T(1)\Delta_T(-1) \neq 0$ is isomorphic to the Witt-equivalence group of matrices A satisfying $(A - {}^tA)(A + {}^tA)$ is non-singular. With this isomorphism we associate the isometric structure $\big(A + {}^tA, -A^{-1}\,{}^tA\big)$ to a square matrix A over $\mathbf{Q}$ such that $(A - {}^tA)(A + {}^tA)$ is non-singular. This isomorphism allows to study Witt-equivalence of matrices in terms of cobordism classes of isometric structures.

Let $\big(<,>, T\big)$ be an isometric over $\mathbf{Q}$. Let $\Lambda = \mathbf{Q}[t, t^{-1}]$ be the ring of Laurent polynomials over $\mathbf{Q}$. We consider V the vector space on which $<,>$ and T are defined as a Λ-module, defining the action of t by T.

Let $\Delta_T(t) = \prod_{i=1}^{r} \lambda_i(t)^{e_i}$ be the factorisation of $\Delta_T(t)$ with irreducible factors over $\mathbf{Q}$. To each irreducible factor λ_i we define $V_{\lambda_i} = \operatorname{Ker} \lambda_i(t)^N$ for N a large integer, such a V_{λ_i} is called a *primary component* of V. Moreover, V is the direct sum $V = \bigoplus_{i=1}^{r} V_{\lambda_i}$.

Let $\lambda(t)$ be a symmetric[1] irreducible factor of Δ_T, then Levine defined

(1) $\varepsilon_\lambda\big(<,>,T\big)$ equals to the exponent of $\lambda(t)$ in $\Delta_T(t)$ mod 2.

(2) $\sigma_\lambda\big(<,>,T\big)$ equals to the signature of the restriction of $<,>$ to V_λ over $\mathbf{R}$.

(3) $\mu_\lambda\big(<,>,T\big) = \big(-1,-1\big)^{\frac{r(r+3}{2}} \big(\det(<,>),-1\big)^r S\big(<,>\big)$
where $<,>$ is of rank $2r$, the Hilbert symbol for $<,>$ over $\mathbf{R}$ is denoted by $\big(,\big)$ and $S\big(<,>\big)$ is the Hasse symbol over Q

We have the following theorem.

Theorem 2.3 ([11]). *Two isometric sturctures α and β are in the same cobordism class if and only if $\varepsilon_\lambda(\alpha) = \varepsilon_\lambda(\beta)$, $\sigma_\lambda(\alpha) = \sigma_\lambda(\beta)$ and $\mu_\lambda(\alpha) = \mu_\lambda(\beta)$ for all $\lambda(t)$ for which these invariants are defined.*

3. Results

3.1. Cobordism classes of Brieskorn knots.

First, we prove the following proposition.

Proposition 3.1. *Let K_P and K_Q be two Brieskorn knots associated with two polynomials P and Q with $n+1$ variables. If there exists two distinct prime numbers p and q such that the spherical Brieskorn knots $K_{P_{p,q}}$ and $K_{Q_{p,q}}$ are not cobordant, then K_P and K_Q are not cobordant.*

Proof. Since the polynomial $P_{p,q}$ is obtained from P by adding a two variables polynomial of the form $z_i^p + z_{i+1}^q$, then according to K. Sakamoto [14] we know that the Seifert form of the Brieskorn knot $K_{P_{p,q}}$ is obtained by the tensor product of the Seifert form of K_P with a square matrix $A_{p,q}$. By Durfee [7] we know that $A_{p,q}$ is a matrix with only 0 and ± 1 coefficents which are only determined by p and q.

If two Brieskorn knots K_P and K_Q are cobordant, then they have witt-equivalent Seifert forms A_{K_P} and A_{K_Q} defined on $\mathbf{Z}$-modules G_P and G_Q. Let M, a submodule of $G_P \oplus G_Q$, be a metabolizer for $A_{K_P} \oplus -A_{K_Q}$ and let d be the dimension of the square matrix $A_{p,q}$. Then the submodule $\bigoplus_{i=1}^{d} M_i$ of $\bigoplus_{i=1}^{d} G_{P,i} \oplus G_{Q,i}$, where M_i is a copy of M and $G_{P,i} \oplus G_{Q,i}$ is a copy of $G_P \oplus G_Q$, is a metabolizer for $\big(A_{K_P} \otimes A_{p,q}\big) \oplus \big(-A_{K_Q} \otimes A_{p,q}\big)$.

Hence the two Seifert forms $A_{K_P} \otimes A_{p,q}$ and $A_{K_Q} \otimes A_{p,q}$ are Witt-equivalent. Since the Brieskorn knots $K_{P_{p,q}}$ and $K_{Q_{p,q}}$ are spherical, we know that they are cobordant. We have proved the proposition by contraposition. □

Proposition 3.2. *The Brieskorn knots K_P and K_Q associated to the polynomials*

$$P(\mathbf{z}) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad Q(\mathbf{z}) = \sum_{j=1}^{n+1} z_j^{b_j}$$

[1]i.e., such that $\lambda(t) = \pm t^{\deg(\lambda)} \lambda(t^{-1})$.

are cobordant if and only if the Brieskorn knots K_{P_+} and K_{Q+} associated to the polynomials

$$P_+(\mathbf{z}) = z_{n+2}^2 + \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad Q_+(\mathbf{z}) = z_{n+2}^2 + \sum_{j=1}^{n+1} z_j^{b_j}$$

are cobordant.

Proof. According to [14] the knots K_P and K_{P_+} have same Seifert forms and the same is true for K_Q and K_{Q_+}. Since the cobordism class of fibered knots is completely determined by the algebraic cobordism class of its Seifert form (see [1]) we have that K_P and K_Q have algebraically cobordant Seifert forms if and only if the same holds for K_{P_+} and K_{Q_+}. Then, the Brieskorn knots K_P and K_Q are cobordant if and only if the Brieskorn knots K_{P_+} and K_{Q+} are cobordant. □

Recall that if A is a Seifert matrix of a Brieskorn knots K_P associated with a Brieskorn polynomial $P(\mathbf{z}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$, then its Alexander polynomials is defined by $\Delta_K(t) = \det\big(t\,A + (-1)^n\,{}^tA\big)$. Since Brieskorn knots are fibered knots, we know that the Alexander polynomial is the characteristic polynomial of the monodromy.

Remark 3.3. The knots K_P and K_{P_+} have same Seifert forms, hence they have same Alexander polynomials.

Proposition 3.4. *The set of irreducible factors of the Alexander polynomial over* $\mathbf{Q}$ *of a Brieskorn knot is an invariant of its cobordism class.*

Proof. By the monodromy theorem proved by E. Brieskorn [5] the Alexander polynomial of Brieskron knots are some products of cyclotomic polynomials. Moreover, Alexander polynomials of Brieskorn knots has been computed by F. Pham [13] and E. Brieskorn [4]. More precisely, let P be the polynomial $P(\mathbf{z}) = \sum_{j=1}^{n+1} z_j^{a_j}$, then Brieskorn gave the following factorization over $\mathbf{C}$ for the alexander polynomial of the Brieskorn knots K_P

$$\Delta_{K_P}(t) = \prod_{0<i_k<a_k} \big(t - \zeta_{a_1}^{i_1}.\cdots.\zeta_{a_n}^{i_n}\big) \qquad (*)$$

where $\zeta_{a_k} = e^{\frac{2\pi i}{a_k}}$.

Let K_P and K_Q be two Brieskorn knots with distinct irreducible factors in their Alexander polynomial, then they have roots aver $\mathbf{C}$ which are distinct. The Brieskorn knots $K_{P_{p,q}}$ and $K_{Q_{p,q}}$ are defined with the same integers p and q which are distinct prime numbers and both are coprime with all exponents of P and Q. Then, according to $(*)$ the Alexander polynomials of $K_{P_{p,q}}$ and $K_{Q_{p,q}}$ have distinct irreducible factors since the do not have the same complex roots.
Moreover, proposition 2.2 implies that $K_{P_{p,q}}$ and $K_{Q_{p,q}}$ are spherical Brieskorn knots.

According to [7], if A is the Seifert form associated to the Milnor fibration, then its intersection form is defined by the relation $S = A + (-1)^n\,{}^tA$ and its monodromy is defined by the relation $h = (-1)^{n-1}A^{-1}\,{}^tA$. Moreover by proposition 3.2 one can suppose that $(-1)^n = 1$. Hence we have that $\big(S, h\big)$ is an isometric structure on $H_n(F, \mathbf{Q})$ where F is the Milnor fiber of the isolated singularity at 0 of a Brieskorn polynomial, S is the intersection form and h is the monodromy. On top of that, up to invertible element, the characteristic polynomial of h is the Alexander polynomial defined using the Seifert form which is a product of symmetric cyclotomic polynomials.

Finally, according to theorem 2.3, the two spherical Brieskorn knots $K_{P_{p,q}}$ and $K_{Q_{p,q}}$ cannot be cobordant since they do not have the same list of invariants. $\square$

In [3] we proved that cobordant Brieskorn knots must have same exponents, up to order, provided no exponent is a multiple of another one. Now, we have the following result which is an immediate corollary of proposition 3.4.

Corollary 3.5. *Let K_P and K_Q be the Brieskorn knots associated to the polynomials*

$$P(\mathbf{z}) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad Q(\mathbf{z}) = \sum_{j=1}^{n+1} z_j^{b_j}.$$

Let $\mathscr{P}_P$ and $\mathscr{P}_Q$ be the sets of distinct irreducible factors of each polynomial P and Q. If $\mathscr{P}_P \neq \mathscr{P}_Q$, then the knots K_P and K_Q are not cobordant.

Proof. Set $\zeta_{a_k} = e^{\frac{2\pi i}{a_k}}$, as before we have the following factorizations

$$\Delta_{K_P}(t) = \prod_{0<i_k<a_k} \left(t - \zeta_{a_1}^{i_1} \cdot \cdots \cdot \zeta_{a_n}^{i_n}\right)$$

and

$$\Delta_{K_Q}(t) = \prod_{0<i_k<b_k} \left(t - \zeta_{b_1}^{i_1} \cdot \cdots \cdot \zeta_{b_n}^{i_n}\right).$$

When $\mathscr{P}_P \neq \mathscr{P}_Q$ the polynomials Δ_{K_P} and Δ_{K_Q} admit distinct complex roots. Hence they do not have the same irreducible factors over $\mathbf{Q}$. By proposition 3.4 the knots K_P and K_Q are not cobordant. $\square$

In the following, for each Brieskorn knot, we will define a list of integers which only depends on its exponents. Since these integers are related to the factorization of the Alexander polynomial of as a product of cyclotomic polynomials, then we will get an invariant of cobordism classes of Brieskorn knots.

In the following, when α and β are two integers, we denote by $\alpha \wedge \beta$ the greatest common divisor of α and β and we denote by $[\alpha, \beta]$ the lowest common multiple of α and β.

Let k and d be two integers. Set $\mu_k(d)$ be the greatest divisor of $k \wedge d$ which is coprime with $\frac{k}{k \wedge d}$.

Then, we associate to the couple of integer (k, d) a set of integers defined as follows

$$\Psi_k(d) = \begin{cases} \left\{ \frac{[k,d]}{l}, \ \text{ if } l \text{ is a divisor of } \frac{k\,\mu_k(d)}{k \wedge d} \right\} \setminus \{k\} & \text{if } \mu_k(d) \leq 2 \\ \\ \left\{ \frac{[k,d]}{l}, \ \text{ if } l \text{ is a divisor of } \frac{k\,\mu_k(d)}{k \wedge d} \right\} & \text{else} \end{cases}$$

Definition 3.6. When K_P is a Brieskorn knot associated with the polynomial $P(z) = \sum_{j=1}^{n+1} z_j^{a_j}$, for all distinct prime numbers p and q which are both coprime to each $a_1, \ldots, a_{n+1}$ we define the set $\Xi_{K_P,p,q}$ as follows.

(1) $\Xi^0_{K_P,p,q} = \{pq\}$
(2) for $i = 1$ to $n+1$ we set $\Xi^i_{K_P,p,q} = \left\{\Psi_{\alpha_j}(a_i) \mid \alpha_j \in \Xi^{i-1}_{K_P,p,q}\right\}$
(3) $\Xi_{K_P,p,q} = \Xi^{n+1}_{K_P,p,q}$

We will prove that $\Xi_{K_P,p,q}$ is invariant in a cobordism class of Brieskorn knots.

Proposition 3.7. *Let K_P be a Brieskorn knot associated with the polynomial $P(\mathbf{z}) = \sum_{j=1}^{n+1} z_j^{a_j}$. Let p and q be two distinct prime numbers which are both coprime to $a_1, \ldots, a_{n+1}$, then the set of integers $\Xi_{K_P,p,q}$ is an invariant of its cobordism class*

Proof. When $P(\mathbf{z})$ is a Brieskorn polynomial with $\mathbf{z} = (z_1, z_2, \ldots, z_{n+1})$ and $n \geq 1$; I. Savel'ev [15] compute the Alexander polynomial of $Q(\mathbf{z}, z_{n+2}) = P(\mathbf{z}) + z^d_{n+2}$ where d is an integer.

Let $\Phi_n(t)$ be the n-th cyclotomic polynomial. If the Alexander polynomial of $P(\mathbf{z})$ is

$$\Delta_P(t) = \prod_{l=1}^{N} \Phi_{k_l}(t)^{\tau_l}, \text{ with } \tau_l > 0,$$

then

$$\Delta_Q(t) = \prod_{l=1}^{N}\Big(\prod_{\nu\mid\lambda_{k_l,d}} \frac{\Phi_{\frac{[k_l,d]}{\nu}}(t)^{\varphi(\mu_{k_l}(d))}}{\Phi_{k_l}(t)}\Big)^{\tau_l} \qquad (\star)$$

where $\lambda_{k_l,d} = \frac{d\mu_{k_l}(d)}{k_l \wedge d}$. We see that $\{\Phi_{k_1}, \ldots, \Phi_{k_N}\}$ is the set of irreducible factors of $\Delta_P(t)$ and $\left\{\Phi_{\Psi_{\alpha_j}(d)} \mid \alpha_j \in \{k_1, \ldots, k_N\}\right\}$ is the set of irreducible factors of $\Delta_Q(t)$.

According to [4], the Alexander polynomial of the Brieskorn knot $K_{x^p+y^q}$ is

$$\Delta_{p,q}(t) = \frac{(t^{\frac{pq}{r}} - 1)^r (t-1)}{(t^p - 1)(t^q - 1)}$$

where $r = p \wedge q$. If p and q are distinct prime numbers, then we have

$$\Delta_{p,q}(t) = \frac{\Phi_{pq}(t)\Phi_p(t)\Phi_q(t)\Phi_1(t)^2}{\Phi_p(t)\Phi_q(t)\Phi_1(t)^2} = \Phi_{pq}(t).$$

When $\mu_{k_l}(d) \leq 2$, then $\varphi(\mu_{k_l}(d))$ equals 1 and Φ_{k_l} is no longer a factor of Δ_Q in $(\star)$. Hence $\Xi_{K_P,p,q}$ is the set of irreducible factors of $\Delta_{P_{p,q}}$. According to proposition 3.4, when p and q are two distinct prime numbers which are both coprime to each $a_1, \ldots, a_{n+1}$, the set $\Xi_{K_P,p,q}$ is an invariant of the cobordism class of K_P. □

Remark 3.8. The last proposition is just a reformulation of proposition 3.4. But it gives a computable list of integers which is an invariant of the cobordism class of a Brieskorn knot.

3.2. **Examples.** Recall that Alexander polynomials of cobordant knots must satisfy a Fox-Milnor relation. In [2] we only use this property to determine if knots cannot be cobordant. Moreover, in the same paper, example 3.8 gave the examples of Brieskorn knots K_f and K_g where $n > 3$, $p_1, p_2, \ldots, p_{n-3} \geq 2$ and

$$f(\mathbf{z}) = z_1^{p_1} + z_2^{p_2} + \cdots + z_{n-3}^{p_{n-3}} + z_{n-2}^8 + z_{n-1}^8 + z_n^4 + z_{n+1}^4$$

and

$$g(\mathbf{z}) = z_1^{p_1} + z_2^{p_2} + \cdots + z_{n-3}^{p_{n-3}} + z_{n-2}^6 + z_{n-1}^6 + z_n^6 + z_{n+1}^6$$

for wich it was unknown if they are in the same cobordism class since the product of the Alexander polynomials fulfills a Fox-Milnor relation.

With proposition 3.4 we have a new method to determine if such Brieskorn knots lies in the same cobordism class or not. More precisely, if 3 is coprime with all $p_1, \ldots, p_{n-3}$, then the set of prime factors of the exponents of the polynomials f and g are distinct.

Recall that when $P(\mathbf{z}) = \sum_{j=1}^{n+1} z_j^{a_j}$ we have $\Delta_{K_P}(t) = \prod_{0<i_k<a_k} \big(t - \zeta_{a_1}^{i_1} \cdot \cdots \cdot \zeta_{a_n}^{i_n}\big)$. Hence, the polynomials Δ_{K_f} and Δ_{K_g} have not the same set of complex roots. By proposition 3.4 we see that in this case the Brieskorn knots K_f and K_g are not cobordant.

3.3. **Conjecture.** We found more examples of Brieskorn knots for which distinct exponents imply distinct cobordism classes. Moreover, according to proposition 3.7 it seems difficult to find examples of Brieskorn knots which are cobordant and have distinct exponents up to order. Hence, we formulate the following conjecture.

Conjecture 3.9. *The Brieskorn knots associated to the polynomials*

$$P(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad Q(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

are cobordant if and only if $a_j = b_j$, $j = 1, 2, \ldots, n+1$, *up to order.*

Remark that if this conjecture is true, then the multiplicity of a Brieskorn knot will be an invariant of its cobordism class.

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