The pre-Lie relation, rooted trees and the Magnus expansion

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Motivation

1) initial value problem I

$$\dot{f}(t) = a(t)f(t), \qquad f(0) = 1$$

$$I(f)(t) := \int_0^t f(s)ds \qquad f = 1 + I(af)$$

$$f = 1 + \sum_{n>0} \underbrace{I\Big(aI(\cdots I(aI}_{n\text{-times}}(a))\cdots)\Big)$$

$$a \prec f := aI(f) \qquad f = 1 + I\Big(a + \sum_{n>1} \underbrace{a \prec (\cdots \prec (a \prec}_{n\text{-times}} a)\cdots)\Big)$$

$$\dot{f} = a + a \prec \dot{f}$$

$$(a \prec b) \prec c = aI(b)I(c) = aI\Big(I(b)c + b\,I(c)\Big) = a \prec \Big(b \succ c + b \prec c\Big)$$

2) Bogoliubov's counterterm recursion:

H Hopf algebra of Feynman graphs, $\phi, \phi_- : H \to A$

$$\phi_- = \mathbf{1} - R(\phi_- \star \rho), \qquad \rho = \phi - \mathbf{1}$$

$$R(\phi) \star R(\delta) = R\Big(R(\phi) \star \delta + \phi \star R(\delta)\Big) - R(\phi \star \delta)$$

$$\phi_- = \mathbf{1} + \sum_{n>0} (-1)^n \underbrace{R\Big(R(\cdots R(\rho)}_{n\text{-times}} \star \rho) \cdots \star \rho\Big)$$

$\psi \succ \phi := R(\psi) \star \phi$

$$\phi_- = \mathbf{1} - R\Big(\rho - \sum_{n>1} (-1)^n \Big(\cdots \underbrace{(\rho \succ \rho) \succ \rho \cdots\Big) \succ \rho}_{n\text{-times}}\Big)$$

$$\bar{\phi} = \rho - \rho \succ \bar{\phi}$$

$$R(\phi) \star R(\delta) = R\Big(R(\phi) \star \delta + \phi \star R(\delta)\Big) - R(\phi \star \delta)$$

$$\begin{aligned}
\phi \succ (\tau \succ \mu) &= \Big(R(\phi) \star R(\tau)\Big) \star \mu \\
&= R\Big(R(\phi) \star \tau + \phi \star R(\tau) - \phi \star \tau\Big) \star \mu \\
&= \Big(\phi \succ \tau + \phi \prec \tau\Big) \succ \mu
\end{aligned}$$

$$\phi \prec \rho := -\phi \star \tilde{R}(\rho), \qquad \tilde{R} = (\mathrm{id} - R)$$

$$\begin{aligned}
(\phi \prec \tau) \prec \mu &= \phi \star \Big(\tilde{R}(\tau) \star \tilde{R}(\mu)\Big) \\
&= \phi \star \tilde{R}\Big(\tilde{R}(\tau) \star \mu + \tau \star \tilde{R}(\mu) - \tau \star \mu\Big) \\
&= \phi \prec \Big(\tau \succ \mu + \tau \prec \mu\Big)
\end{aligned}$$

$$(\phi \succ \tau) \prec \mu = -R(\phi) \star \tau \star \tilde{R}(\mu) = \phi \succ (\tau \prec \mu)$$

3) initial value problem II

$$\dot{Y} = [A, Y], \qquad Y(0) = Y_0$$

$$\dot{Y} = AY - YB$$

$$\dot{Y} = C - \dot{Y} \succ B + A \prec \dot{Y}, \qquad C := AY_0 - Y_0B$$

$$Y = XY_0Z^{-1}$$

$$X = \exp\big(\Omega(A)\big), \qquad \dot{X} = AX, \quad X(0) = 1$$

$$Z = \exp\big(\Omega(B)\big), \qquad \dot{Z} = BZ, \quad Z(0) = 1$$

W. Magnus: [1954] Magnus expansion ($ad_A[B] := [A, B]$)

$$\boxed{\dot{\Omega}(A) = \sum_{m \geq 0} \frac{B_m}{m!}\, ad^{(m)}_{\Omega(A)}[A]}.$$

Bernoulli numbers: $\sum_{n \geq 0} \frac{B_m}{m!} z^m = \frac{z}{\exp(z)-1} = 1 - \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \cdots$

$$\Omega(A)(t) \;=\; \int_0^t A(s_1)ds_1 - \frac{1}{2}\int_0^t \Big[\Omega(A)(s_1), A(s_1)\Big]ds_1 + \frac{1}{12}\int_0^t \Big[\Omega(A)(s_1), [\Omega(A)(s_1), A(s_1)]\Big]ds_1 + \cdots$$

$$\Omega(A) = \sum_{n>0} \Omega_n(A)$$

$$\Omega(A)(t) \;=\; \underbrace{\int_0^t A(s_1)ds_1}_{\Omega_1(A)(t)} \; \underbrace{-\frac{1}{2}\int_0^t \Big[\int_0^{s_1} A(s_2)ds_2, A(s_1)\Big]ds_1}_{\Omega_2(A)(t)}$$

$$+\frac{1}{4}\int_0^t \Big[\int_0^{s_1} \Big[\int_0^{s_2} A(s_3)ds_3, A(s_2)\Big]ds_2, A(s_1)\Big]ds_1$$

$$+\frac{1}{12}\int_0^t \Big[\int_0^{s_1} A(s_3)ds_3, \Big[\int_0^{s_1} A(s_2)ds_2, A(s_1)\Big]\Big]ds_1 + \cdots$$

Rota–Baxter Algebra

Definition: $\theta \in \mathbb{K}$, (A, R), $R : A \to A$

$$R(x)R(y) = R\Big(R(x)y + xR(y)\Big) + \theta R(xy)$$

- The map $\tilde{R} := -\theta\mathrm{id} - R$ is a Rota–Baxter map.

Examples:

Riemann integral: $I(f)(t) := \int_0^t f(s)ds$

Projectors: Let $A = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$, $\sum_{n=-k}^{\infty} a_n \varepsilon^n$, $\theta = -1$

$$R\Big(\sum_{n=-k}^{\infty} a_n \varepsilon^n\Big) := \sum_{n=-k}^{-1} a_n \varepsilon^n.$$

q-difference equation: $\sigma_q f(x) := f(qx)$

$$\partial_q f(x) := (id - \sigma_q) f(x) = f(x) - f(qx)$$

$$\widehat{P}_q[f] := \sum_{n \geq 0} \sigma_q^n f$$

$$\widehat{P}_q[f]\widehat{P}_q[g] = \widehat{P}_q\Big[\widehat{P}_q[f]\ g\Big] + \widehat{P}_q\Big[f\ \widehat{P}_q[g]\Big] - \widehat{P}_q\Big[fg\Big]$$

Summation:

$$S[f](x) := \sum_{n > 0} f(x+n)$$

$$S[f]S[g] = S\Big[S[f]\ g\Big] + S\Big[f\ S[g]\Big] + S\Big[fg\Big]$$

Dendriform Algebra

$(A, \prec, \succ)$

$$
\begin{aligned}
(a \prec b) \prec c &= a \prec (b \prec c + b \succ c) \\
(a \succ b) \prec c &= a \succ (b \prec c) \\
a \succ (b \succ c) &= (a \prec b + a \succ b) \succ c
\end{aligned}
$$

associative product:

$$a * b := a \prec b + a \succ b$$

left pre-Lie and right pre-Lie product:

$$a \rhd b := a \succ b - b \prec a \qquad a \lhd b := a \prec b - b \succ a$$

$$(a \rhd b) \rhd c - a \rhd (b \rhd c) = (b \rhd a) \rhd c - b \rhd (a \rhd c)$$

$$(a \lhd b) \lhd c - a \lhd (b \lhd c) = (a \lhd c) \lhd b - a \lhd (c \lhd b)$$

$$[a, b] := a * b - b * a = a \rhd b - b \rhd a = a \lhd b - b \lhd a$$

Dendriform algebras from RB operators

$(A, \succ, \prec)$

$$a \succ b \;:= R(a)b \qquad\qquad a \prec b \;:= -a\tilde{R}(b)$$

$$(a \prec b) \prec c \;=\; a \prec (b * c)$$

$$a\tilde{R}(b)\tilde{R}(c) = -a\tilde{R}\Big(R(b)c - b\tilde{R}(c)\Big)$$

$$(a \succ b) \prec c \;=\; a \succ (b \prec c)$$

$$-\Big(R(a)b\Big)\tilde{R}(c) = -R(a)\Big(b\tilde{R}(c)\Big)$$

$$a \succ (b \succ c) \;=\; (a * b) \succ c$$

$$R(a)R(b)c = R\Big(R(a)b - b\tilde{R}(a)\Big)c$$

associative prod. $\quad a * b \;=\; a \succ b + a \prec b = R(a)b - a\tilde{R}(b)$

pre-Lie prod. $\quad a \rhd_R b \;=\; a \succ b - b \prec a = R(a)b + b\tilde{R}(a)$

$(\overline{A}, \prec, \succ)$

$$a \prec \mathbf{1} := a =: \mathbf{1} \succ a \qquad \mathbf{1} \prec a := 0 =: a \succ \mathbf{1},$$

$a * \mathbf{1} = \mathbf{1} * a = a$ and $\mathbf{1} * \mathbf{1} = \mathbf{1}$, but $\mathbf{1} \prec \mathbf{1}$ and $\mathbf{1} \succ \mathbf{1}$ are not defined.

$$\exp^*(x) := \sum_{n \geq 0} x^{*n}/n! \qquad \text{resp.} \qquad \log^*(\mathbf{1} + x) := -\sum_{n > 0} (-1)^n x^{*n}/n$$

$$\begin{aligned} w_{\succ}^{(n)}(x_1, \ldots, x_n) &:= \Big(\cdots (x_1 \succ x_2) \succ x_3 \cdots \Big) \succ x_n \\ w_{\prec}^{(n)}(x_1, \ldots, x_n) &:= x_1 \prec \Big(x_2 \prec \cdots (x_{n-1} \prec x_n) \cdots \Big) \end{aligned}$$

$$\begin{aligned} \ell^{(n)}(x_1, \ldots, x_n) &:= \Big(\cdots (x_1 \rhd x_2) \rhd x_3 \cdots \Big) \rhd x_n \\ r^{(n)}(x_1, \ldots, x_n) &:= x_1 \lhd \Big(x_2 \lhd \cdots (x_{n-1} \lhd x_n) \cdots \Big) \end{aligned}$$

Dendriform Equations I

- equation of degree (m,n) in $\overline{A}[[\lambda]]$:

$$U = a + \sum_{q=1}^{m} \lambda^q \omega_{\succ}^{(q+1)}(U, b_{q1}, \dots, b_{qq}) + \sum_{p=1}^{n} \lambda^p \omega_{\prec}^{(p+1)}(c_{p1}, \dots, c_{pp}, U)$$

- equations of degree $(0,1)$ and $(1,0)$:

$$Y = a + \lambda b \prec Y \qquad \text{resp.} \qquad Z = c + \lambda Z \succ d$$

- equation of degree $(1,1)$:

$$X = a + \lambda X \succ b + \lambda c \prec X$$

- pre-Lie equation: $c = -b$

$$X = a + \lambda X \rhd b$$

Let $\overline{A}$ be a unital dendriform algebra. Let $Y = Y(c)$ and $Z = Z(b)$, $b, c \in A$ be solutions of in $\overline{A}[[\lambda]]$ to:

$$Y = \mathbf{1} + \lambda c \prec Y \qquad \text{resp.} \qquad Z = \mathbf{1} + \lambda Z \succ b$$

Proposition The product $X = Y * Z$ gives a solution to the degree $(1,1)$ equation for the particular case $a = \mathbf{1} \in \overline{A}$

$$X = \mathbf{1} + \lambda X \succ b + \lambda c \prec X$$

$$\tilde{Y} = \mathbf{1} - \lambda \tilde{Y} \succ c \qquad \text{and} \qquad \tilde{Z} = \mathbf{1} - \lambda b \prec \tilde{Z}$$

$$\tilde{Y} * Y = \mathbf{1} = Y * \tilde{Y} \qquad \text{resp.} \qquad \tilde{Z} * Z = \mathbf{1} = Z * \tilde{Z}$$

Proposition For $a, c \in A$ and $Y = Y(b)$ and $Z = Z(d)$, the equations:

$$V = a + \lambda b \prec V \qquad \text{resp.} \qquad W = c + \lambda W \succ d$$

are solved by

$$V = Y * \left(Y^{-1} \succ a\right) \qquad \text{and} \qquad W = \left(c \prec Z^{-1}\right) * Z$$

Theorem Let $Y = Y(c)$ and $Z = Z(b)$, $c, b \in A$.

$$X = a + \lambda X \succ b + \lambda c \prec X$$

of degree $(1, 1)$ and $a \in A$ is solved by:

$$X = Y * \left(Y^{-1} \succ a \prec Z^{-1}\right) * Z$$

Corollary Let $Z = Z(b)$ be a solution to $Z = \mathbf{1} + \lambda Z \succ b$. The solution to the left pre-Lie equation:

$$X = a + \lambda X \rhd b,$$

is given by:

$$X = Z^{-1} * \left(Z \succ a \prec Z^{-1}\right) * Z$$

$$X = Ad^*_{Z^{-1}} \circ \Theta_Z(a)$$

$\Theta_Z(a) := Z \succ a \prec Z^{-1}$

$\Theta_A \circ \Theta_B(x) = \Theta_{A*B}(x)$

$\dot{Y}=[A,Y]$,
$Y(0)=Y_0$

$$Y' = C + Y' \prec A - A \succ Y' = C - Y' \triangleright A$$

$\dot{X}=AX$
$X(0)=\mathbf{1}$

$$X' = \mathbf{1} + A \prec X'$$

$$Y' = Ad^*_{X'^{-1}} \circ \Theta_{X'}(C)$$

dendriform algebra coming from RB operator R:

$$\begin{aligned} Y = R(Y') &= R(Ad^*_{X'^{-1}} \circ \Theta_{X'}(C)) \\ &= R(X'^{-1})R\Big(X' \succ a \prec X'^{-1}\Big)R(X') \end{aligned}$$

$$R(a * b) = R(a \succ b + a \prec b) = R(R(a)b - a\tilde{R}(b)) = R(a)R(b)$$

Pre-Lie Magnus expansion

Theorem: Let $(\bar{A}, \prec, \succ)$ be a dendriform algebra. Let $\Omega' := \Omega'(\lambda a)$, $a \in A$, be the element of $\lambda A[[\lambda]]$ such that

$$Z = \exp^*(\Omega')$$

where Z is the solution of $Z = 1 - Z \succ \lambda a$. This element obeys the following recursive equation:

$$\Omega'(\lambda a) = \frac{L_{\rhd}[\Omega']}{\exp(L_{\rhd}[\Omega'])-1}(\lambda a) = \sum_{m\geq 0} \frac{B_m}{m!} \, L_{\rhd}^{(m)}[\Omega'](\lambda a)$$

where the B_l's are the Bernoulli numbers.

$$L_{\rhd}[a](b) \; := \; a \rhd b = a \succ b - b \prec a$$

$$Y = 1 + \lambda a \prec Y, \qquad Y = \exp^*(-\Omega')$$

Classical Magnus expansion:

$$A \prec B(x) := A(x) \cdot \int_0^x B(t)dt \qquad A \succ B(x) := \int_0^x A(t)dt \cdot B(x)$$

$$A \succ B - B \prec A = A \rhd_I B = [I(A), B] \quad -\theta BA$$

$$\begin{aligned}
\dot{\Omega} &= \sum_{m \geq 0} \frac{B_m}{m!} \, ad^{(m)}_{I(\dot{\Omega}(A))}[A] \\
&= A - \frac{1}{2}[I(\dot{\Omega}), A] + \frac{1}{12}\Big[I(\dot{\Omega}), [I(\dot{\Omega}), A]\Big] + \sum_{m \geq 4} \frac{B_m}{m!} \, ad^{(m)}_{I(\dot{\Omega}(A))}(A) \\
&= A - \frac{1}{2}\dot{\Omega} \rhd_I A + \frac{1}{12}\dot{\Omega} \rhd_I (\dot{\Omega} \rhd_I A) + \sum_{m \geq 4} \frac{B_m}{m!} \, ad^{(m)}_{I(\dot{\Omega})}(A) \\
&= \sum_{m \geq 0} \frac{B_m}{m!} L^{(m)}_{\rhd_I}[\dot{\Omega}](A)
\end{aligned}$$

$$L_{\rhd_I}[a](b) \;\; := \;\; a \rhd_I b = I(a)b - bI(a)$$

Dendriform Equations II

• equation of degree $(m, 0)$ $((0, n))$ in $\overline{A}[[\lambda]]$:

$$X = a + \sum_{q=1}^{m} \lambda^q \omega_{\succ}^{(q+1)}(X, b_{q1}, \ldots, b_{qq})$$

Let $(A, \prec, \succ)$ be a dendriform algebra. Then the space $\mathcal{M}_n(A)$ of square $n \times n$-matrices with entries in A is a dendriform algebra, with operations defined by:

$$(a \prec b)_{ij} := \sum_{i=1}^{n} a_{ik} \prec b_{kj}, \qquad (a \succ b)_{ij} := \sum_{i=1}^{n} a_{ik} \succ b_{kj}$$

$\overline{\mathcal{M}_n(A)} := \mathcal{M}_n(\bar{A})$ is the matrix dendriform algebra augmented with a unit, identified with the diagonal matrix $\mathbf{1}_2$ with dendriform units $\mathbf{1}$'s on the diagonal and 0 elsewhere.

• equation of degree $(2,0)$:

$$X = d + \lambda X \succ c + \lambda^2 (X \succ b) \succ a$$

Proposition Let $\Omega' = \log^* Z \in \mathcal{M}_2(A[[\lambda]])$, where Z is the solution of the dendriform matrix equation of degree $(1,0)$:

$$Z = \mathbf{1}_2 + \lambda Z \succ \begin{pmatrix} c & b \\ a & 0 \end{pmatrix}$$

given by the pre-Lie Magnus expansion:

$$\Omega'(\lambda M) = \lambda M - \frac{1}{2}\lambda^2 M \rhd M + \frac{1}{4}\lambda^3 (M \rhd M) \rhd M + \lambda^3 \frac{1}{12} M \rhd (M \rhd M) \cdots$$

Then the solution X for $d = \mathbf{1}$, is such that the line vector $(X,\, \lambda X \succ b)$ is the first line of the matrix $Z = \exp^*(\Omega')$, i.e.:

$$X = (1,0) Z (1,0)^{\mathrm{t}}.$$

• equation of degree $(m,0)$:

$$X = a_{00} + \sum_{q=1}^{m} \lambda^q \omega_{\succ}^{(q+1)}(X, a_{q1}, \ldots, a_{qq})$$

matrix of size $N := 1 + \frac{m(m-1)}{2}$:

$$M_m = \begin{pmatrix} M_{m-1} & A_1 \\ A_2 & A_3 \end{pmatrix} = \left(\begin{array}{cc} M_{m-1} & \begin{pmatrix} a_{m1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{mm} & 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & a_{m2} & 0 & \cdots & 0 \\ 0 & 0 & a_{m3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{m(m-1)} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \right)$$

$$Z = \mathbf{1}_N + Z \succ M_m.$$

Rooted trees and the coefficients in the pre-Lie Magnus expansion

$$\Omega'(\lambda a) = \sum_{n\geq 0} \frac{B_n}{n!} L_{\triangleright}^{(m)}[\Omega'](\lambda a).$$

$$\begin{aligned}\Omega'(\lambda a) &= \lambda a - \lambda^2 \frac{1}{2} a \triangleright a + \lambda^3 \frac{1}{4}(a \triangleright a) \triangleright a + \lambda^3 \frac{1}{12} a \triangleright (a \triangleright a) \\ &- \lambda^4 \frac{1}{8}((a \triangleright a) \triangleright a) \triangleright a - \lambda^4 \frac{1}{24}\Big((a \triangleright (a \triangleright a)) \triangleright a \\ &+ a \triangleright ((a \triangleright a) \triangleright a) + (a \triangleright a) \triangleright (a \triangleright a)\Big) + \dots\end{aligned}$$

planar rooted trees $\mathcal{T}_{pl}$: $t = B_+(t_1, \cdots, t_n)$

$$\alpha(t) := \frac{B_n}{n!} \prod_{i=1}^{n} \alpha(t_i) = \prod_{v \in V(t)} \frac{B_{f(v)}}{f(v)!}.$$

$ker(\alpha) = \{t \in \mathcal{T}_{pl} \,|\, \exists v \in V(t) : f(v) = 2n+1, n > 0\}$

$$r_{\triangleright}^{(n)}(a_1,\ldots,a_n) := a_1 \triangleright \Big(a_2 \triangleright \Big(a_3 \triangleright \cdots (a_{n-1} \triangleright a_n)\Big)\cdots\Big)$$

$F[\bullet](a) = a$ and for $t = B_+(t_1 \cdots t_n)$, $n \geq 1$:

$$\begin{aligned} F[t](a) &:= r_{\triangleright}^{(n+1)}\Big(F[t_1](a),\ldots,F[t_n](a),a\Big) \\ &= F[t_1](a) \triangleright \Big(F[t_2](a) \triangleright \cdots (F[t_n](a) \triangleright a)\Big) \end{aligned}$$

Theorem: The pre-Lie Magnus expansion can be written:

$$\Omega'(\lambda a) = \sum_{t \in \mathcal{T}_{pl}^{e1}} \alpha(t) F[t](\lambda a)$$

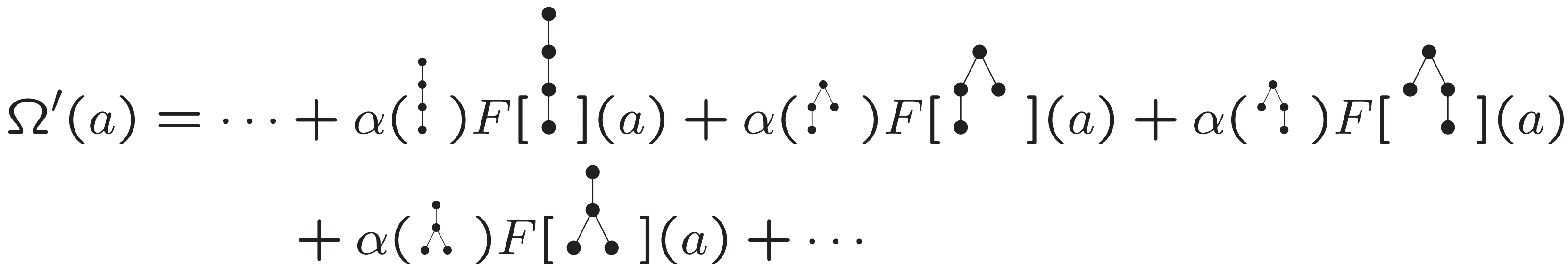
$T(z) = \sum_{k \geq 0} T_k z^k$ be the Poincaré series of $\mathcal{T}_{pl}^{e1}$

$$T(z) - z^2 T(z)^3 = \frac{1}{1-z}$$

which in turn yields the recursive definition of the coefficients T_n:

$$T_n = 1 + \sum_{\substack{p,q,r \geq 0 \\ p+q+r=n-2}} T_p T_q T_r.$$

1, 1, 2, 4, 10, 26, 73, 211, 630, ... ([A049130])



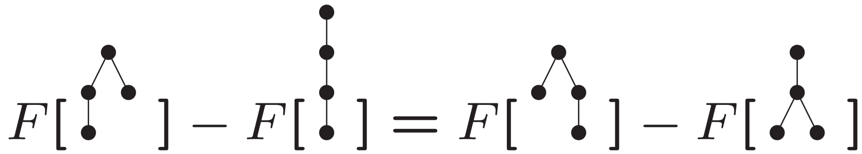
pre-Lie relation: $t = B_+(t_1 \cdots t_n)$

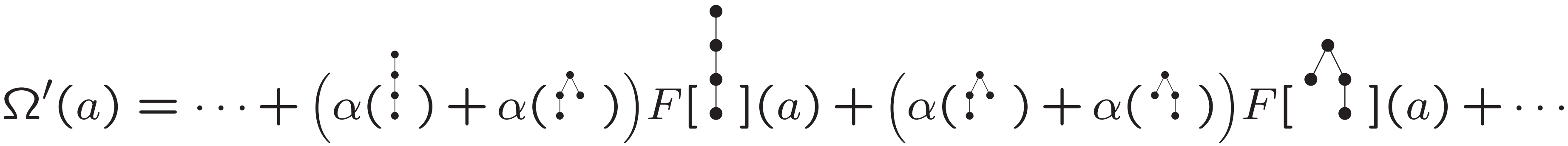
$$F\Big[B_+(t_1 t_2 \cdots t_n)\Big] - F\Big[B_+((t_1 \bullet t_2) \cdots t_n)\Big] =$$
$$F\Big[B_+(t_2 t_1 \cdots t_n)\Big] - F\Big[B_+((t_2 \bullet t_1) \cdots t_n)\Big]$$

Butcher product: t_1, $t_2 = B_+(t_2^1 \cdots t_2^n)$

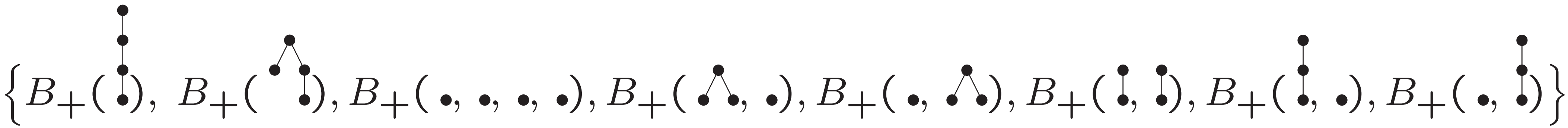
$$t_1 \bullet t_2 := B_+(t_1 \; t_2^1 \cdots t_2^n)$$

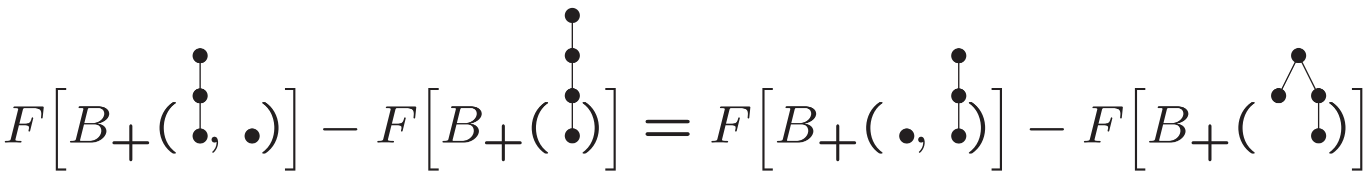
reduced order 4:





order 5:





$$((a \triangleright a) \triangleright a) \triangleright (a \triangleright a) - \Big(((a \triangleright a) \triangleright a) \triangleright a\Big) \triangleright a$$
$$= a \triangleright \Big(((a \triangleright a) \triangleright a) \triangleright a\Big) - \Big(a \triangleright ((a \triangleright a) \triangleright a)\Big) \triangleright a$$

THANK YOU!!