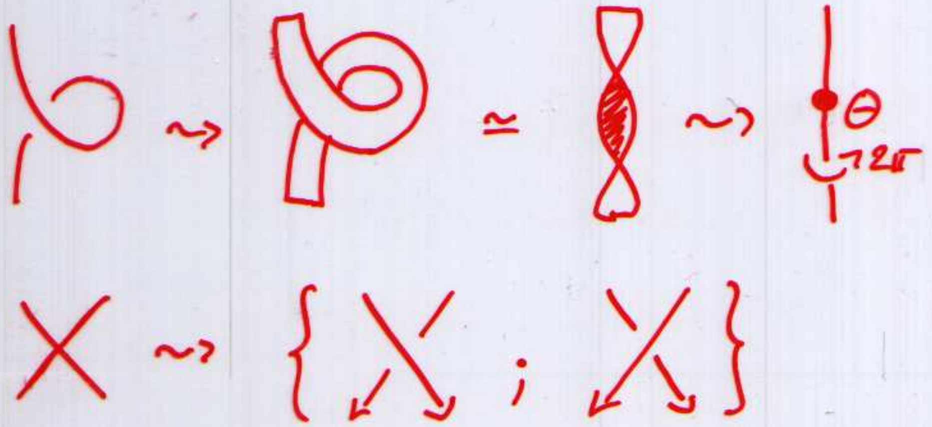
On the ribbon Hopf algebra structure of plethystically generated subcharacter rings of $GL(\infty)$



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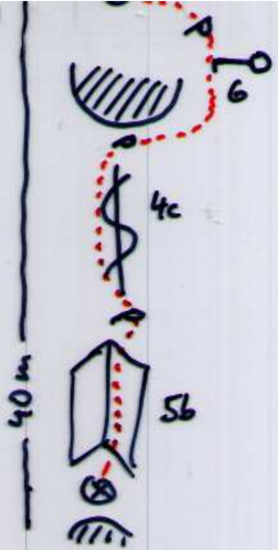


topos of our route

$$
\begin{tikzcd}
GL(\infty) \arrow[rr, two heads] \arrow[dd, hook] \arrow[dr, two heads] & & H_\pi(\infty) \arrow[dd, hook] \arrow[dr, two heads] & \\
& GL(N) \arrow[rr, two heads, crossing over] \arrow[dd, hook] & & H_\pi(N) \arrow[dd, hook] \\
Rep(GL(\infty)) \arrow[rr, tail, two heads] \arrow[dd, two heads] \arrow[dr, two heads] & & Rep(H_\pi(\infty)) \arrow[dd, two heads] \arrow[dr, two heads] & \\
& Rep(GL(N)) \arrow[rr, tail, two heads, crossing over] \arrow[dd] & & Rep(H_\pi(N)) \arrow[dd] \\
Inv(GL(\infty)) \arrow[rr, tail, two heads] \arrow[dr, two heads] & & Inv(H_\pi(\infty)) \arrow[dr, two heads] & \\
& Inv(GL(N)) \arrow[rr, tail, two heads] & & Inv(H_\pi(\infty))
\end{tikzcd}
$$

$\approx Symm(\infty)$

$\approx$ PSH on one generator

$\approx Symm(N)$

Note: as HA we have

$$Symm(\infty) \cong Inv(GL(\infty)) \cong Inv(H_\pi(\infty))$$

Note: We need far more structure as the HA provides, such as ONB, duality, Schur positivity, ... so that the enriched structures encode different settings.

Symm(∞) (coproduct part, on alphabets)

Arithmetic Hierarchy

$f \in \mathrm{Symm}(\infty)$ $\quad X, Y, \ldots$ ∞-alphabets

'+' : $f(X) \xrightarrow{\Delta} f(X+Y) = \sum_{(f)} f_{(1)}(X) f_{(2)}(Y)$

outer coproduct

'$*$' : $f(X) \xrightarrow{\delta} f(XY) = \sum_{[f]} f_{[1]}(X) f_{[2]}(Y)$

inner coproduct

'$\wedge$' : $f(X) \xrightarrow{\mathring{\Delta}} f(\text{"}X^{Y}\text{"}) = \sum_{\langle f \rangle} f_{\langle 1 \rangle}(X) f_{\langle 2 \rangle}(Y)$

Brouder-Schmitt convention on Sweedler indices $\Delta \to ()$, $\delta \to []$, $\mathring{\Delta} \to \langle\rangle$

plethysm coproduct

[non-cocommutative, non-linear (polynomial), cf. Connes-Moscovici, Connes-Kreimer non-lin. coprod..]

For some (Schur) basis we have:

$\Delta s_\lambda = \sum c^\lambda_{\mu\nu}\, s_\mu \otimes s_\nu$ $\qquad c^\lambda_{\mu\nu}$: Littlewood-Richardson-rule

$\delta s_\lambda = \sum g^\lambda_{\mu\nu}\, s_\mu \otimes s_\nu$ $\qquad g^\lambda_{\mu\nu}$: Murnaghan-Nakayama-rule

$\mathring{\Delta} s_\lambda = \sum p^\lambda_{\mu\nu}\, s_\mu \otimes s_\nu$ $\qquad p^\lambda_{\mu\nu}$: "coplethysm"

$\mathrm{Inv}(GL(\infty)) \simeq \mathrm{Symm}(\infty)$ 'irreducibles'

complete symmetric functions:

$$\mathcal{H}(z) := \sum_{n\geq 0} h_n(x)\, z^n := \prod_{i=1}^{\infty} \frac{1}{1 - x_i z}$$

$$h_\lambda(x) = h_{\lambda_1}(x)\, h_{\lambda_2}(x) \ldots h_{\lambda_\ell}(x) \qquad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)_{\geq}$$

integer partition

monomial symmetric functions:

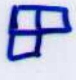
$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \qquad \alpha \models m \quad \text{integer composition}$$

$$m_\alpha(X) = \mathrm{sym}(x^\alpha)$$

Schur functions:

m = card of alphabet X

$$s_\lambda^{[m]}(X) = \sum_{\alpha \in SSYT(\lambda, m)} x^\alpha \qquad + \text{ 'stability'}$$

Ex: $\lambda = (21) =$  $\quad m = 3 \rightarrow X = x_1 + x_2 + x_3$

$$s_{21}^{[3]}(X) \cong \begin{smallmatrix}1&1\\2\end{smallmatrix} + \begin{smallmatrix}1&1\\3\end{smallmatrix} + \begin{smallmatrix}1&2\\2\end{smallmatrix} + \begin{smallmatrix}1&2\\3\end{smallmatrix} + \begin{smallmatrix}1&3\\2\end{smallmatrix} + \begin{smallmatrix}1&3\\3\end{smallmatrix} + \begin{smallmatrix}2&2\\3\end{smallmatrix} + \begin{smallmatrix}2&3\\3\end{smallmatrix}$$

$$\cong x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2\, x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

Orthogonality:

Schur functions will be identified as characters of irred. $GL(N)$ representations and are hence assumed to be orthogonal:

Schur-Hall scalar product $\langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}$

GL(N) - characters

$V = \mathrm{span}_{\mathbb{C}}(\{x_i\}_{i=1}^N) \in \mathrm{Fd\,Vect}_{\mathbb{C}}$

$GL(N) \simeq \mathrm{Aut}(V) \subset \mathrm{End}(V) \simeq \mathrm{Hom}(V,V)$

$g \cdot v = gv$ action of $g \in GL(N)$ on $v \in V$

$V \hookrightarrow T(V) \xrightarrow{\pi} S(V)$

diagonal action

$g \cdot T^k(V) \;:\; g \cdot (v_1 \dots v_k) = (gv_1, gv_2, \dots, gv_k)$

$h \in GL(N \cdot k)$ acting on $W \equiv T^k(V)$

$\boxed{\mathrm{Cent}_{GL(N\cdot k)}(GL(N)) = S_k}$ $\qquad S_k$: sym grp on k letters

Schur-Weyl-duality

Schur $\Rightarrow$ S_k - character labels (integer partitions) specify the $GL(N)$ irreducible modules

$T(V) = \bigoplus_{k \geq 0} T^k(V) \qquad T^k(V) = \bigoplus_{\lambda \vdash k} V^\lambda$

where V^λ is an irreducible vectorspace of Schur symmetry type 'λ'.

characters (equi-spectral classes)

$g \sim h \iff \mathrm{inv}(g) = \mathrm{inv}(h)$ as multisets

An equivalence class $[g]$ is characterized by its spectral data (eigenvalues) $\{x_i\}_{i=1}^{N}$ (related to the Cartan subgroup)

$$V^\lambda \xrightarrow{\mathbb{T}} [V^\lambda] \cong s_\lambda(x_1 \dots x_N)$$

$\boxed{\text{we drop } [\dots] \text{ from now onwards}}$

Schur functions => Schur functors

$$V \xrightarrow{s_\lambda} V^\lambda$$

$(x_1 \dots x_N) \qquad s_\lambda(x_1 \dots x_N)$

functoriality: plethysm $\cong$ composition

$$\begin{tikzcd} V \arrow[r, "s_\mu"] \arrow[rr, bend right, "{s_\lambda[s_\mu] \simeq \{\mu\} \otimes \{\lambda\}}"'] & V^\mu \equiv W \arrow[r, "s_\lambda"] & W^\lambda = (V^\mu)^\lambda \end{tikzcd}$$

composition — plethysm notation of Littlewood $\{\lambda\} \equiv s_\lambda$

$\{s_\lambda\}_{\lambda \vdash n, n \geq 0}$ constitute a set of polynomial functors on $\mathrm{FdVect}_{\mathbb{C}}$, ... see Macdonalds book.

Graphical description:

$$V \otimes V \xrightarrow{\langle s_\mu, s_\nu \rangle} V^\mu \otimes V^\nu \xrightarrow{\ \cong\ } \bigoplus_\lambda \oplus^{c^\lambda_{\mu\nu}} V^\lambda$$

$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$ Littlewood – Richardson rule

$$V \otimes V \xrightarrow{\langle s_\mu, s_\nu \rangle} V^\mu \otimes V^\nu \xrightarrow{\ \cong\ } \bigoplus_\lambda g^\lambda_{\mu\nu} V^\lambda$$

$s_\mu * s_\nu = \sum_\lambda g^\lambda_{\mu\nu} s_\lambda$ Murnaghan – Nakayama rule

$$V \xrightarrow{s_\mu} V^\mu \cong W \xrightarrow{s_\nu} W^\nu \cong (V^\mu)^\nu \longrightarrow \bigoplus_\lambda p^\lambda_{\mu\nu} V^\lambda$$

$s_\nu[s_\mu] = \sum_\lambda p^\lambda_{\mu\nu} s_\lambda$ plethysm $\simeq$ composition

this can be recast in a λ-ring structure

- the coefficients $c^\lambda_{\mu\nu}$, $g^\lambda_{\mu\nu}$, $p^\lambda_{\mu\nu}$ are defined via characteristic polynomials, see for example D. Knutson

- linear substitutions:

$$\begin{array}{ccc} U & \xrightarrow{s_\lambda} & U^\lambda \\ \downarrow & & \downarrow \\ V & \xrightarrow{s_\lambda} & V^\lambda \end{array}$$

Subgroup Characters

$$\begin{array}{ccccc} GL & \longrightarrow & \mathrm{Inv}(GL) & \simeq & \mathrm{Symm}\ HA \\ \downarrow & & \downarrow \simeq & & \\ H_\pi & \longrightarrow & \mathrm{Inv}(H_\pi) & \simeq & \mathrm{Symm}\ HA \end{array}$$

Def: $H^\pi = \{g \mid g \in GL ;\ g \circ J^\pi = J^\pi\}$

Note: If $H_\pi \in \{ GL(N-1), O(N), Sp(N)\}$ 'classical groups' then one still finds a basis of irreducible charaters, otherwise only indecomposables occure.

Ex: $\text{mod-}\mathrm{Inv}(GL) = \mathrm{span}_{\mathbb{C}} \{ s_\lambda \}_{\lambda \vdash n, n=1}^{\infty}$

$\text{mod-}\mathrm{Inv}(O) = \mathrm{span}_{\mathbb{C}} \{ o_\lambda \}_{\lambda \vdash n, n=1}^{\infty}$

iso: $s_\lambda \xrightarrow{\simeq} o_\lambda$ as modules

Note: this map does not respect the algebraic structure (product, coproduct) of Symm

Q: How does a GL-character $s_\lambda \equiv \{\lambda\}$ decompose into O-characters $o_\lambda \equiv [\lambda]$?

Ex: $J^{ij} = J^{(ij)} + J^{[ij]} \simeq \{2\} + \{11\}$

$\simeq \square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix}$

$GL \downarrow O \quad J^{(ij)} = \mathring{J}^{(ij)} + J g^{(ij)} \qquad \{2\} = [2] + [0]$

$J^{[ij]} = \mathring{J}^{[ij]} \qquad \{11\} = [11]$

in general:

$$\{\lambda\} = \sum_{\mu} d_{\lambda\mu} [\mu]$$

'Wick type' contraction, how?

- irred. characters are orthogonal

  $\langle \mid \rangle : \mathrm{Symm} \times \mathrm{Symm} \to \mathbb{Z}$

  $\langle s_\lambda \mid s_\mu \rangle = \delta_{\lambda\mu}$

- Schur function series (Littlewood, King, Wybourne, ...)

  $M(z) = \sum h_n(x) z^n = \sum \{n\} z^n = \prod_i \frac{1}{1-x_i z}$

- Actions:
  - $M(z)\, s_\lambda = \sum s_{(n)} \cdot s_\lambda\, z^n$ multiplicative
  - $M^\perp(z)\, s_\lambda = \sum s_{\lambda/s_{(n)}}\, z^n$ as differential op.

  $\langle s_\lambda^\perp s_\mu \mid s_\nu \rangle := \langle s_\mu \mid s_\lambda \cdot s_\nu \rangle$

Now: $GL(N) \downarrow GL(N-1)$ $\quad s_\lambda^{[N]} \to s_\lambda^{[N-1]}$

$$s_\lambda \mapsto M^\perp \cdot s_\lambda = \langle M(z) \mid s_{\lambda(1)} \rangle\, s_{\lambda(2)} \Big|_{z=1}$$

$$\{n\} \mapsto \{n\}/\{r\} = \{n\} + \{n-1\} + \ldots + \{0\}$$

iso since

$$M(z)^{-1} = L(z) = M[-s_1](z) = \sum (-)^n e_n z^n \qquad e_n = s_{(1^n)}$$

graphical:

branchings, cont. ~

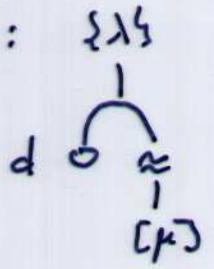
$Gl \downarrow O$:

$$D(z) = M[s_2](z) = \prod_{i\geq j} \frac{1}{1 - x_i x_j z}$$

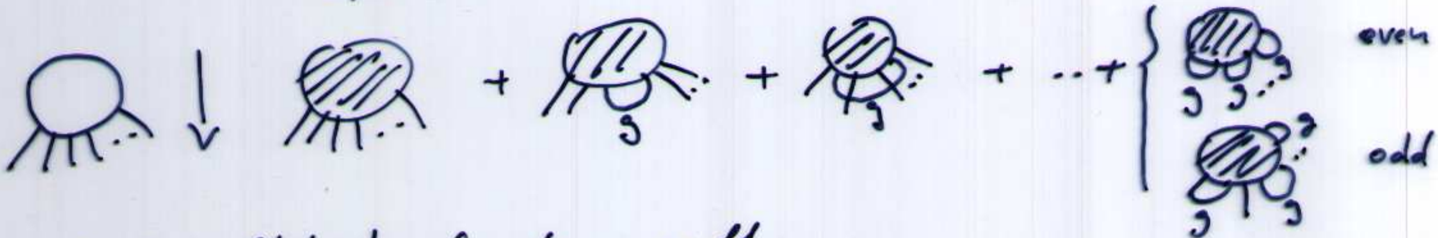
$$C(z) = L[s_2](z) = \prod_{i\geq j} 1 - x_i x_j z = M[-s_2](z)$$

- iso $s_\lambda \simeq o_\lambda \quad \leftrightarrow \quad \{\lambda\} \simeq [\lambda]$
- $\{\lambda\}\downarrow [\mu] = [\lambda/_D] = \langle D \mid s_{\lambda(1)} \rangle \tilde{s}_{\lambda(2)} = \langle D \mid s_{\lambda(1)} \rangle o_{\lambda(2)}$

graphical:



$$d(x) = \langle D \mid x \rangle = \langle M[s_2] \mid x \rangle$$



product: Clebsch - Gordan coeff.

$$[\mu]\cdot[\nu] = \bigoplus \tilde{c}^{\lambda}_{\mu\nu} \, [\lambda]$$

$$= \sum_{\zeta} [\mu/_\zeta \cdot \nu/_\zeta] \qquad \text{Newell - Littlewood}$$

$$[\mu]\cdot[\nu] = [(\{\mu\}/_C \cdot \{\nu\}/_C)/_D]$$

$$= [(\{\mu\}/_{M[-s_2]} \cdot \{\nu\}/_{M[-s_2]})/_{M[s_2]}]$$

$$= c(\{\mu_{(3)}\})\, c(\{\nu_{(3)}\})\, d(\{\mu_{(2)}\nu_{(2)}\}) \;\; [\{\mu_{(1)} \cdot \nu_{(1)}\}]$$

$$= \underbrace{(\partial d)(\{\mu_{(1)}\}, \{\nu_{(1)}\})}_{\text{coupl.}} \;\; [\{\mu_{(2)}\}\{\nu_{(2)}\}]$$

generalize: $C = M[s_2] = L[s_2]$
$D = M[-s_2]$ $\qquad s_2 \simeq [J^{(ij)}]$

to plethystic Schur function series

$$M_\pi = M[s_\pi] \qquad L_\pi = M_\pi^{-1} = M[-s_\pi]$$

Thm [F Jarvis King Wybourne]

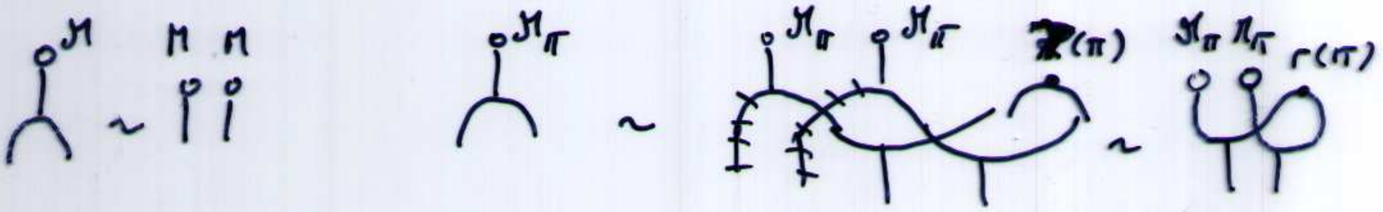
$$((\mu))_\pi \, ((\nu))_\pi = \bigoplus (\tau^\pi)^\lambda_{\mu\nu} \, ((\lambda))_\pi$$

with the key lemma:

$$\Delta M[s_\pi] = (\Delta M)[s_\pi] = (M \otimes M)[s_\pi]$$
$$= M_\pi \otimes M_\pi \cdot \mathcal{R}(\pi)$$

where
- $\mathcal{R}(\pi)$ is an universal R-matrix
- $\mathcal{R}(\pi)$ depends on inner products and plethysms

Note: $\Delta M = M \otimes M$ was a group like series



$r : \Lambda \to \mathrm{Symm} \otimes \mathrm{Symm}$ is a coquasitriangular structure

Char - GL $\simeq$ Char - O

While combinatorially very different (module + basis) the HA of characters are isomorphic, however, spoiling for example the λ-ring structure

Weyl groups: Schur-Weyl duality $GL \leftrightarrow S_N$ $O \leftrightarrow D_N$

inv. S_n poly: $\sigma \in S_n$

$$\sigma \cdot (x_1 + \ldots + x_n) = x_1 + \ldots + x_n = p_1(x)$$

$$\sigma \cdot (x_1^2 + \ldots + x_n^2) = x_1^2 + \ldots + x_n^2 = p_2(x)$$

Adams operations

$$\vdots$$

$$\sigma(x_1^n + \ldots + x_n^n) = x_1^n + \ldots + x_n^n = p_n(x)$$

$$\sigma(x_1^{n+i} + \ldots + x_n^{n+i}) = x_1^{n+i} + \ldots + x_n^{n+i} = p_{n+i}(x)$$

but $p_{n+i}(x) = f_i(p_1(x) \ldots p_n(x))$ $\forall i \geq 1$

D_N: $y_i = x_i^2$ and look at sym. functions in the y_i.

$D_N = \{\tau_1\} \cup \{\sigma_i\}_{i=1}^N$ where $S_n = \{\sigma_i\}_{i=1}^N$ $\quad \tau_1 x_1 = -x_1$

$$\hookrightarrow p_1(y) = y_1 + \ldots + y_n = x_1^2 + \ldots + x_n^2$$

$$p_2(y) = y_1^2 + \ldots + y_n^2 = x_1^4 + \ldots + x_n^4$$

$$\vdots$$

results: [F. Jarvis King]

i) Symm $\simeq$ Char-GL $\simeq$ Char-O $\simeq$ Char-Sp

and conjecturally $\simeq$ Char-H_π

ii) while Char-GL is a self dual HA
the othe HA's are <u>not</u> self dual

- dual HA's are in general no longer build own finite modules

- <u>each</u> Char-H_π comes with an <u>own</u> Schur-Hall scalar product on irreducibles (indecomposables) making these orthonormal

$$\langle o_\lambda \mid o_\mu \rangle_2 = \delta_{\lambda\mu}$$

$$\langle sp_\lambda \mid sp_\mu \rangle_{11} = \delta_{\lambda\mu}$$

$$\vdots$$

iii) there are isomorphic (but combinatorially demanding) versions of sym. fct. bases

s_λ, e_λ, h_λ, f_λ, m_λ, p_λ
<u>and</u> new bases
o_λ, sp_λ, πs_λ

The ribbon graph HA structure

- Inv (GL) $\rightarrow$ $V \simeq V^*$ (as iso classes) $\sim \downarrow \simeq \uparrow$
- we need Schur-positivity to detect proper characters
- $\mathrm{Symm} \simeq \{\lambda\}_{\lambda \vdash n, n=0}^{\infty} \simeq ((\lambda))_{\pi,\, \lambda \vdash n, n=0}^{\infty}$ as modules

ribbons:

blackboard framing

this introduces a ribbon element and

a twist 

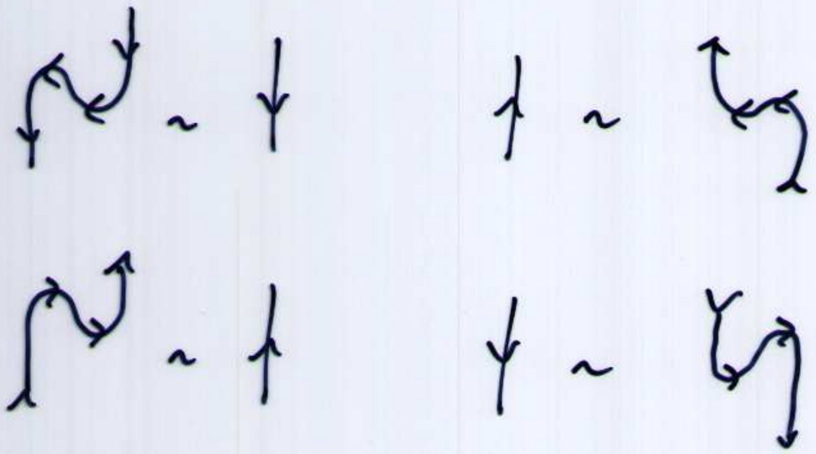
the ribbon structure relates left-to-right and right-to-left duality (closed structures)

closure

$I \xrightarrow{b_V} V \otimes V^*$ $\qquad$ $V^* \otimes V \xrightarrow{d_V} I$

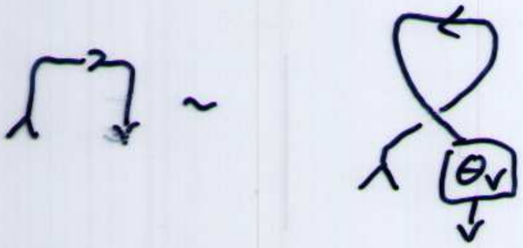

$I \xrightarrow{b'_V} V^* \otimes V$ $\qquad$ $V \otimes V^* \xrightarrow{d_V} I$


with: (e.g. Kassel XIV.3 (3.4) (3.5))



Now:

c_{V,V^*}



c_{V,V^*}

„Quantum" Trace and Dim

Let f ~  be a morphism on Symm

Def: Tr_V^{π} : $\mathrm{Hom}(V^{\otimes}, V^{\otimes}) \longrightarrow \mathrm{End}(I)$

$Tr_V^{\pi}(f) =$  $=$  c_{V,V^*}

this trace fulfills:

i) $Tr_V^{\pi}(fg) = Tr_V^{\pi}(gf)$

ii) $Tr_V^{\pi}(f \otimes g) = Tr_V^{\pi}(f)\, Tr_V^{\pi}(g)$

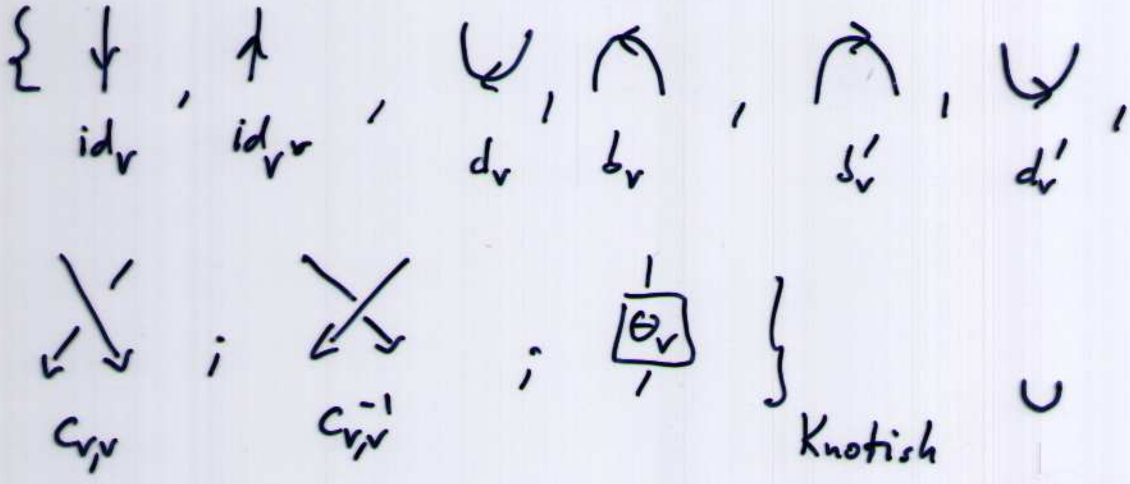
iii) $Tr_V^{\pi}(f^*) = Tr_V^{\pi}(f)$ in $\mathrm{End}(I)$

 ~ 

Def: $\dim^{\pi}(V) := Tr_V^{\pi}(id_V)$

ribbon HA structure of Char - H_Π

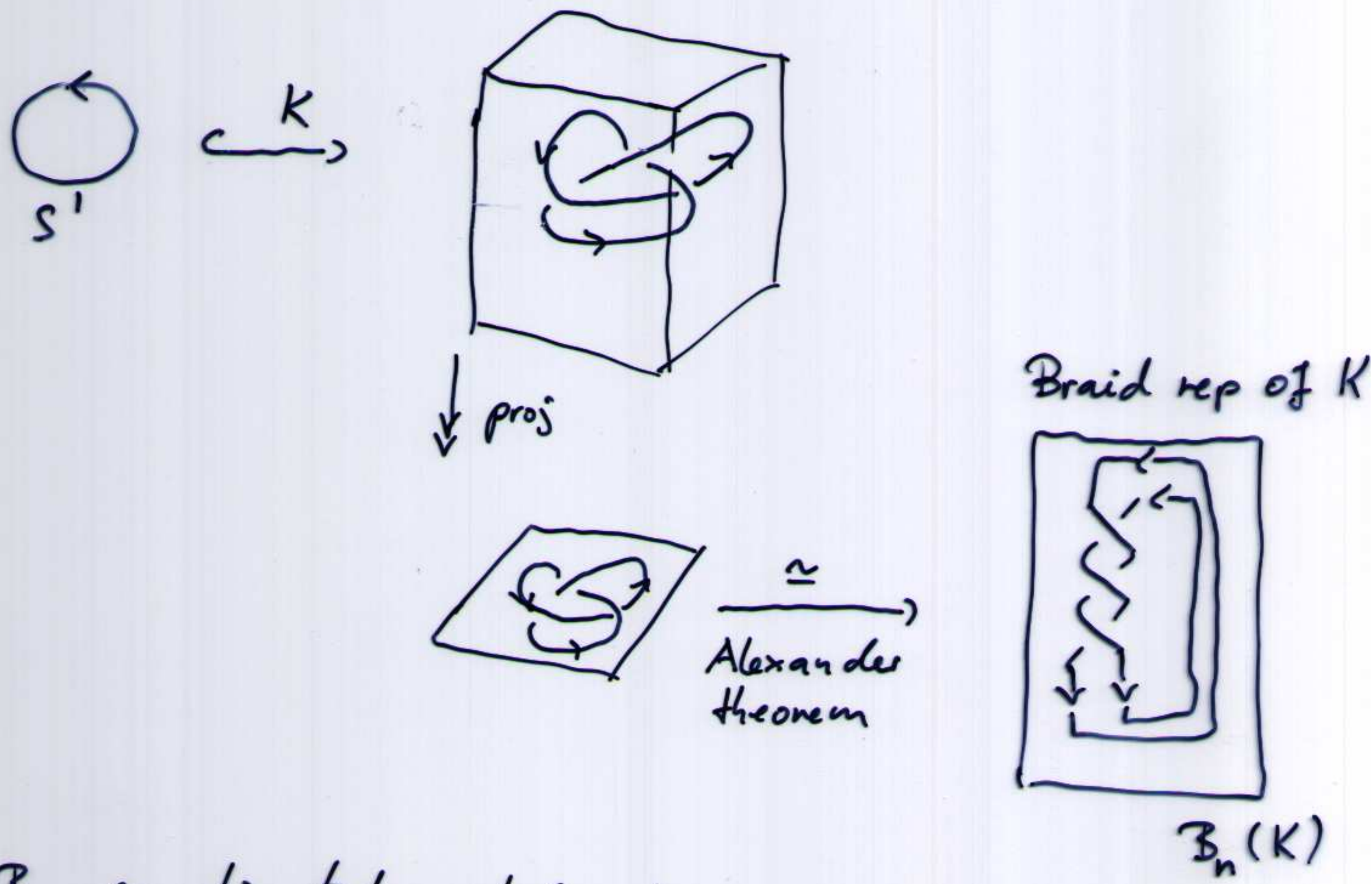
$GL \downarrow H_\Pi$ gives explicit combinatorial expr. for



One gets a Drinfeld double

$\mathcal{D}(H) = H \otimes H^*$ with $H^\vee \neq H$ in general

Knot / Link Invariants



B_n complicated, study homomorphisms

- $B_n \twoheadrightarrow S_n$

  $\{\tau_i\} \longmapsto \{\sigma_i\}$ $\quad \sigma_i^2 = 1 \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

  $\Rightarrow$ trivializes all traces

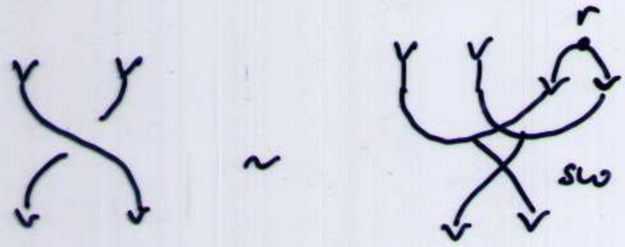
- $B_n \twoheadrightarrow S_n^\pi$ Weyl group of H_π

  allows a non-trivial braiding $c_{V,V}$

  induced by $\mathcal{R}(\pi)$ R-matrix

– preliminary results

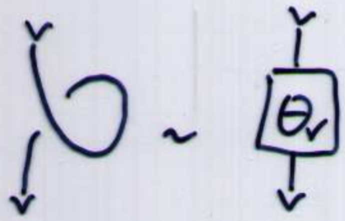
GL ↓ O (GL ↓ Sp)



in Char-O   Char-GL

$\propto \sum_\lambda s_\lambda \otimes s_\lambda$

Cauchy-Kernel



$\Theta_v \propto \sum_\alpha s_\alpha \cdot s_\alpha$

$$\Theta_v = \mathfrak{M}[p_1 p_1] = ((m \circ \delta)\mathfrak{M})[p_1 p_1]$$
$$= \mathfrak{M}_{[1]}[p_1] \cdot \mathfrak{M}_{[2]}[p_2] = \sum_\alpha s_\alpha \cdot s_\alpha$$

<u>But</u>: GL ↓ O is a triangular case and the explicite calculated traces are trivial (i.e. reproduce the dim, etc.)

<u>Way out:</u> Consider $H_\mu \downarrow H_\nu$ for $|\mu| \geq 2$ yields non-trivial (not triangular) R-matrices

-> under investigation.

Landscape picture of structure

Arithmetic hierarchy ->

complexity ↓	'+'	'*'	'^'
QFT	Wick, time-ordered products, operator products	?	renormalization Connes-Kreimer type HA
[Rep(G)]	HA of rep. functions		
Inv(G)	outer HA of sym. funct.	inner bialg. of sym. funct	plethysm and plethysm coproduct
iso ↕			
QM	alg. of creation and annihilation op.	?	?
dim ↓			
Dirichlet conv. of arithmetic functions (Number Theory)	additive arithmetic functions	multiplicative arithmetic functions μ-Möbius as antipode etc.	Hecke operators L-functions and automorphic forms

More layers possible: Quantum protocolls, quantum computing, knots/links, quantum chemistry

Some Rotaism

$$\begin{tikzcd} & \mathbb{N} \arrow[r, "\sigma"] \arrow[d, "x"] & \mathbb{N} \arrow[d, "x"] \\ O \arrow[ur, "0"] \arrow[r, "x_0"'] & \mathbb{N} \arrow[r, "\sigma"] & \mathbb{N} \end{tikzcd}$$

Natural Number Object NNO

$\sigma : n \to n+1$ successor map

captures the theory of recursion

- $m = \sigma^m(0)$ evaluation of $(\sigma \circ .. \circ \sigma)$ on σ

iterate:

- $m + n = \sigma^m(0) '+' \sigma^n(0) = \sigma^{n+m}(0)$
- $m \cdot n = \sigma^n(0) '+' \dots '+' \sigma^n(0) = (\sigma^n)^m(0)$

⋮

do this on a <u>one letter alphabet</u> a by forming symmetric algebras

$\mathbb{k} \xrightarrow{i} \mathbb{k}[a]$ 'adjoining an invariant'

$$\begin{tikzcd} \mathrm{Rings} \arrow[loop left] \arrow[r, bend left, "i"] & \mathrm{Rings}_\bullet \arrow[l, bend left, "\mathrm{eval}"] \arrow[loop right] \end{tikzcd}$$

adjoint functors

$$\begin{tikzcd} \mathbb{k} \arrow[r, bend left, "i"] & (\mathbb{k}[x], x) \arrow[l, bend left, "\mathrm{eval}"] \end{tikzcd}$$

<u>and</u> iterate!

(Remark: Rota-Stein used super symmetric alphabets capturing Ext, Symm and other cases ...)

Sleth (H) of Rota – Stein

(Tens [Tens[B]$^+$] in Bronder – Schmidt)

- $\mathbb{k} \simeq \mathbb{Z} \longrightarrow S[a] \simeq \mathbb{Z}[a]$ (is a HA)

$\mathbb{Z}[a]^+ = \mathrm{span}_{\mathbb{Z}} \{ a^n \}_{n \geq 1}$ underlying module (actually comodule)

prod: $a^n \cdot a^m = a^{n+m}$

coprod: $\Delta a^n = \sum_{n_1+n_2=n} \binom{n}{n_1} a^{n_1} \otimes a^{n_2}$

unit: $1 \to \eta(1) = a^0 = 1 \in \mathbb{Z}[a]$

counit: $\mathbb{Z}[a] \xrightarrow{\varepsilon} \mathbb{Z} \quad \varepsilon(a^n) = \delta_{n0}$

- iterate on the (co)module $\mathbb{k}^+[a]$; $\mathrm{span}_{\mathbb{Z}}\{a^n\}_{n=1}^{\infty} = V$

$\mathcal{D}iv[V] = \mathcal{D}iv[\mathrm{mod}\text{-}\mathbb{k}^+[a]] \simeq \mathbb{Z}^{()}[(a^1), (a^2), (a^3), \ldots]$

note: $S(V)^* \simeq \mathcal{D}iv(V^*)$

$\mathcal{D}iv[V] \ni m_\lambda = (a^1)^{(r_1)} (a^2)^{(r_2)} \cdots (a^n)^{(r_n)}$

$\lambda = (\lambda_1, \lambda_2, \ldots \lambda_\ell) = [1^{r_1} 2^{r_2} 3^{r_3} \ldots n^{r_n}]$

the product in $\mathcal{D}iv[V]$ is not the product of symmetric functions, but up to the binomial factors concatenation (unordered)

$$(a^i)^{(n_i)} \cdot (a^j)^{(n_j)} = \begin{cases} \binom{n_i+n_j}{n_i} (a^i)^{(n_i+n_j)} & i=j \\ (a^i)^{(n_i)} (a^j)^{(n_j)} & \text{else} \end{cases}$$

<u>re</u>introduce the product structure on V

$V = \mathbb{Z}[a]^+$ using a HA twist via an (algebra valued) Laplace pairing:

<u>Def</u>: $\langle\, , \rangle : Div[V] \times Div[V] \longrightarrow Div[V]$

i) $\langle 1, 1 \rangle = 1$

ii) $\langle (a^n)^{(m)} \mid (a^r)^{(s)} \rangle = \delta_{ms}\, (a^{n+r})^{(s)}$

iii) $\langle A \mid BC \rangle = \langle A_{(1)} \mid B \rangle \langle A_{(2)} \mid C \rangle$

$\langle AB \mid C \rangle = \langle A \mid C_{(1)} \rangle \langle B \mid C_{(2)} \rangle$

iv) $\langle A \mid B \rangle = 0$ if $\deg(A) \neq \deg(B)$

<u>Def</u>: $Pleth(S[a])$ as the HA deformation of $Div[V]$ with new product

$$Pleth(V) \simeq (Div[V], \circ)$$

$$A \circ B = \langle A_{(1)} \mid B_{(1)} \rangle \cdot A_{(2)} \cdot B_{(2)}$$

<u>Ex</u>: $m_3 \cdot m_2 = m_{32}$ $\qquad m_2 \cdot m_2 = \binom{2}{1} m_{22}$

$m_3 \circ m_2 = m_5 + m_{32}$

$m_2 \circ m_2 = m_4 + 2\, m_{22}$

showing that $\boxed{Pleth(S[a]) \simeq Symm} \simeq \mathbb{Z}[a^1, a^2, \ldots]^{S_\infty}$ is the HA of sym. functions

$\Rightarrow$ $\circ$-product captures the LRR (in monomial basis) <u>without</u> obscure algebro-comb. structure

Comments and Literature (added after the talk)

The talk touches, but not covers, research done over the last 10 years. I would like to thank my collaborators and coauthors

- Rafal Ablamowicz, TNTech Cookeville, TN, USA
- Christian Brouder, Univ. Paris 6, Paris, FR
- Alesandra Frabetti, Univ. Lyon 1, Lyon, FR
- Peter Jarvis, Univ. Hobart, TAS, Au
- Ronald C. King, Univ. of Southampton, Southampton, UK
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- Brian G. Wybourne, † Univ. Torun, Torun, PL

own work is available from
http://clifford.physik.uni-konstanz.de/~fauser

- Plethystic Hopf algebras are due to G.-C. Rota and Joel A. Stein, PNAS 91, 1994, 13057 ff
- The idea of iteration of structures can neatly be found in F.W. Lawvere, R Rosebrugh, "Sets for Mathematics" Cambridge Univ. Press 2005 but seems to be folklore for category theorists
- Several topics, like conformal field theory, vertex operators, etc are not in the landscape Picture but can be added smoothly.

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