# Power series of non linear operators, effective actions and some combinatorial illustrations

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*Conférence Algèbre combinatoire et Arbres*
*Lyon, May 2008*

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**Feynman diagrams and iterations of effective actions**

- Some properties of the Tutte polynomial
- Loop decomposition of the Symanzik polynomial

# Perturbative solution of a differential equation

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\frac{d^2x^i}{ds^2} &= \sum_j \frac{\partial X^i}{\partial x^j} X^j \\
\frac{d^3x^i}{ds^3} &= \sum_{j,k} \frac{\partial X^i}{\partial x^j} \frac{\partial X^j}{\partial x^k} X^k + \frac{\partial^2 X^i}{\partial x^j \partial x^k} X^j X^k \\
\frac{d^4x^i}{ds^4} &= \sum_{j,k,l} \frac{\partial X^i}{\partial x^j} \frac{\partial X^j}{\partial x^k} \frac{\partial X^k}{\partial x^l} X^l + 3 \frac{\partial^2 X^i}{\partial x^j \partial x^k} \frac{\partial X^k}{\partial x^l} X^j X^l \\
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Each of these terms correspond to **rooted trees** with various weights.

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$$x(s_1) = x(s_0) + h\sum_{i=1}^{n} b_i X(y_i),$$

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This can be formalized using the **Hopf algebra** of rooted trees.

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- coproduct

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{c \text{ admissible cut}} P_c(t) \otimes R_c(t) \qquad (1)$$

admissible cut : any path from any leaf to the root is cut at most once
$R_c(t)$ = connected component of the root
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$\mathcal{H}_T$ is graded by the number of vertices $|t|$.

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Results also valid for the more general graded and commutative Hopf algebras based on **Feynman diagrams**.

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Smooth map $X$ raised to the power of the tree $t$:

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Composition law (B-series):

$$\Psi_\alpha(X) \circ \Psi_\beta(X) = \Psi_{\beta * \alpha}(X), \quad \alpha, \beta \in G_T.$$

# Geometric series

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If $\alpha$ is the character that takes the value $-1$ on the tree with one vertex and 0 otherwise, $\alpha^{-1} = \alpha \circ S$ takes the value 1 on all trees and defines the **geometric series**

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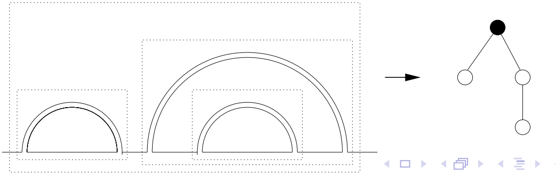
- **resummation** of tree-like structures (example: planar diagrams)



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$I^t_{s,s_0} \subset \mathbb{R}^{|t|}$ is a **treeplex** (generalization of a simplex) obtained by assigning real numbers $s^v$ to the vertices in decreasing order from the root to the leaves, with $s^{\text{root}} \leq s$ and $s^{\text{leaf}} \geq s_0$.

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For a time independent equation we the **tree factorial** $t!$

$$\int_{I^t_{s,s_0}} d^{|t|}s = \frac{1}{t!}.$$

# Geometric interpretation of the coproduct

Comparing both sides of $R_{s_2,s_1} \circ R_{s_1,s_0} = R_{s_2,s_0}$ yields a disjoint union

$$I^{t}_{s_2,s_0} = \bigcup_{c \text{ admissible cut}} \mathfrak{S}_c \left( I^{t'_1}_{s_1,s_0} \times \cdots \times I^{t_n}_{s_1,s_0} \times I^{t''}_{s_2,s_1} \right) \quad (4)$$

with $P_c(t) = t'_1 \dots t'_n$ and $R_c(t) = t''$ and $\mathfrak{S}_c$ a permutation of the labels preserving the ordering of the tree.

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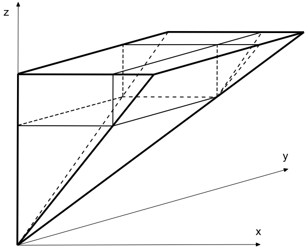
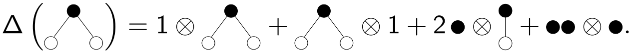
Example:



# The binomial series

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Expanding $(\alpha)^a = (\epsilon + \alpha - \epsilon)^a$ for $\alpha$ the character that takes the value 1 on the tree with one vertex and vanishes otherwise, we obtain the **binomial series** for a non linear operator

$$(\mathrm{id} + X)^a = \sum_t \left( \sum_{n=d_t}^{|t|} N(n,t) \frac{a(a-1)\cdots(a-n+1)}{n!} \right) \frac{X^t}{\mathrm{S}_t}$$

where $N(n,t)$ is the number of surjective maps from the vertices of $t$ to $\{1, \cdots, n\}$, strictly increasing from the root to the leaves (heaps) and $d_t$ is the height of the tree, i.e. the length of the longest path from the root to the leaves.

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**Square root** of a diffeomorphism close to the identity

$$\sqrt{\mathrm{id} + X} = \mathrm{id}\tfrac{1}{2}X - \tfrac{1}{8}X'[X] + \tfrac{1}{16}X'[X'[X]] - \tfrac{5}{128}X'[X'[X'[X]]] \\ + \tfrac{1}{128}X''[X, X'[X]] - \tfrac{1}{2\cdot 64}X'[X''[X,X]] + \tfrac{1}{6\cdot 64}X'''[X,X,X] + \cdots$$

which fulfills $\sqrt{\mathrm{id} + X} \circ \sqrt{\mathrm{id} + X} = \mathrm{id} + X$ up to terms of fifth order.

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If $\beta$ takes the value 1 on all trees, expanding $(\beta)^{-a} = (\epsilon - \alpha)^a$ yields

$$(\mathrm{id} - X)^{-a} = \sum_t \left( \sum_{n=d_t}^{|t|} \widetilde{N}(n,t) \frac{a(a-1)\cdots(a-n+1)}{n!} \right) \frac{X^t}{\mathrm{S}_t}$$

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Equating $(\mathrm{id} - X)^a$ with a binomial series we get identities between $N(n,t)$ and $\widetilde{N}(n,t)$. For instance, for $a = 1, 2$

$$\begin{array}{rcl} 1 & = & \sum_{n=d_t}^{|t|} (-1)^{n+|t|} N(n,t) \\ \widetilde{N}(2,t) & = & \sum_{n=d_t}^{|t|} (-1)^{n+|t|} (n-1) N(t,n) \end{array}$$

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Computing the exponential as $\mathrm{e}^X = \lim_{n\to\infty} \left(\mathrm{id} + \frac{X}{n}\right)^n$ we obtain a combinatorial formula for the **tree factorial**

$$\frac{|t|!}{t!} = N(|t|, t).$$

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$$X[f](s) = sf^2(s) + \int_0^s ds' f^2(s').$$

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Combinatorial intepretation:

$$\frac{1}{|t|!} \prod_v \left(1 + \frac{1}{h_v}\right) = N(t, |t|) \prod_v (1 + h_v)$$
$$= \#\{\text{tree ordered lists of paths from vertices to leaves}\}$$

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Path integral in quantum field theory:

$$\mathcal{Z} = \int [D\phi]\, \mathrm{e}^{-\frac{1}{2}\chi \cdot A_{\Lambda_0}^{-1} \cdot \chi + S_0[\phi]}$$

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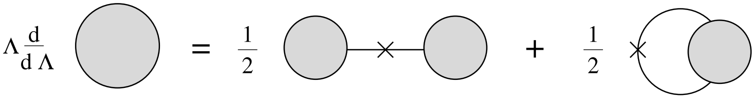
Polchinski's equation:

**Differential equation** for $S_\Lambda$ that generate **Feynman diagrams**



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**Feynman rules** for the perturbed Gaußian integral

$$S'[\phi] = \log\left\{ \int [D\chi]\, \mathrm{e}^{-\frac{1}{2}\chi\cdot A^{-1}\cdot\chi}\ \mathrm{e}^{S[\phi+\chi]} \right\} = \sum_{\gamma \text{ connected diagram}} \frac{A^{\gamma}(S)}{\mathrm{S}_{\gamma}}[\phi]$$

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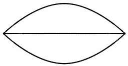
A simple example:



$$\frac{1}{12} \sum_{\substack{i_1, i_2, i_3 \\ j_1, j_2, j_3}} \frac{\partial^3 S}{\partial \phi_{i_1} \partial \phi_{i_2} \partial \phi_{i_3}}[\phi] \, A_{i_1, j_1} A_{i_2, j_2} A_{i_3, j_3} \, \frac{\partial^3 S}{\partial \phi_{j_1} \partial \phi_{j_2} \partial \phi_{j_3}}[\phi]$$

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Composition law analogous to B-series:

$$\underbrace{\Psi_\alpha \circ \Psi_\beta}_{\text{composition}} = \underbrace{\Psi_{\beta * \alpha}}_{\text{convolution}} \qquad \alpha, \beta \in G_F.$$

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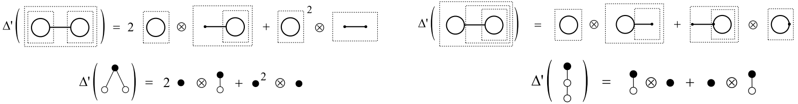

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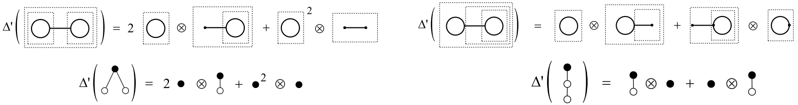

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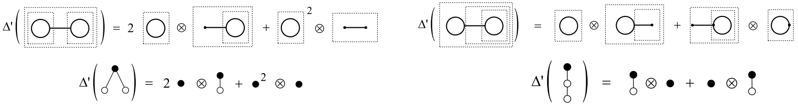

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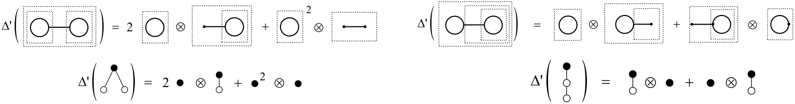

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- diagrams $\rightarrow$ ordered diagrams: $\frac{\gamma}{S_\gamma} \longrightarrow \sum_{\text{orders on } \gamma} \frac{\gamma^{\text{ord}}}{S_\gamma^{\text{ord}}}$.

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In the Lie algebra of $\mathcal{G}_T$ define the infinitesimal characters such that $\delta_{\text{tree}}(\bullet\!\!-\!\!\bullet) = 1$ and $\delta_{\text{loop}}(\bullet\!\!\!\bigcirc) = 1$ and vanish otherwise.

$\alpha = \exp_* \{sa\,\delta_{\text{tree}} + sb\,\delta_{\text{loop}}\}(\gamma) = s^{I_\gamma}\, a^{I_\gamma - L_\gamma}\, b^{L_\gamma}$ obeys the differential equation

$$\frac{d\alpha}{ds} = (a\delta_{\text{tree}} * \alpha + b\delta_{\text{loop}}) * \alpha$$

Any $\delta \in \mathcal{G}_T$ defines two derivations $f \lhd \delta = (\delta \otimes \mathrm{Id}) \circ \Delta(f)$ and $\delta \rhd f = (\mathrm{Id} \otimes \delta) \circ \Delta(f)$.

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# The Tutte polynomial

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The **Tutte polynomial** is a two variable polynomial attached to graphs

$$P_\gamma(x, y) = \sum_{A \subset E} (y-1)^{n(A)} (x-1)^{r(E)-r(A)},$$

where the sum runs over all subsets of the set of edges $E$ of $\gamma$. In the QFT language, the nullity and the rank of a connected diagram can be expressed in terms of the number of internal lines and loops $n(\gamma) = I_\gamma - L_\gamma$ and $r(\gamma) = L_\gamma$.

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Therefore, it can be expressed as the evaluation at $s = 1$ of the character

$$\alpha = \exp_* s\left\{\delta_{\rm tree} + (y-1)\,\delta_{\rm loop}\right\} * \exp_* s\left\{(x-1)\,\delta_{\rm tree} + \delta_{\rm loop}\right\},$$

solution of the **differential equation** with boundary condition $\alpha(0) = \epsilon$,

$$\frac{d\alpha}{ds} = x\,\alpha * \delta_{\rm tree} + y\,\delta_{\rm loop} * \alpha + [\delta_{\rm tree}, \alpha]_* - [\delta_{\rm loop}, \alpha]_*\,.$$

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The character $\beta(\gamma) = s^{l_\gamma}\, Q_\gamma(x, y, a, b)$ obeys the **differential equation**

$$\frac{d\beta}{ds} = x\, \beta * \delta_{\text{tree}} + y\, \delta_{\text{loop}} * \beta + a\, [\delta_{\text{tree}}, \beta]_* - b\, [\delta_{\text{loop}}, \beta]_* \, .$$

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Acting with the automorphism $\varphi_{a^{-1}, b^{-1}}(\gamma) = a^{-(I_\gamma - L_\gamma)}\, b^{-L_\gamma}\, \gamma$, we obtain the **Tutte polynomial differential equation** with modified parameters $\frac{x}{a}$ and $\frac{y}{b}$, so that

$$Q_\gamma(x, y, a, b) = a^{I_\gamma - L_\gamma}\, b^{L_\gamma}\, P_\gamma\left(\frac{x}{a}, \frac{y}{b}\right)$$

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$$S'[\phi] = v \log \left\{ \int [D\chi] \, \mathrm{e}^{-\frac{1}{2v}\chi^2} \; \mathrm{e}^{\frac{S[\phi+\chi]}{v}} \right\}$$

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- Then, we substitute $S'$ into the expression of $S''$ and evaluate at $\psi = 0$,

$$G_{\text{Tutte}}(u, v) = \frac{1}{u} \log I = \frac{1}{u} \log \left\{ \int [D\phi] \, \mathrm{e}^{-\frac{1}{2}\phi^2} \left( \int [D\chi] \, \mathrm{e}^{-\frac{1}{2v}\chi^2} \; \mathrm{e}^{\frac{S[\phi+\chi]}{v}} \right)^q \right\}$$

with $q = uv$.

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When $q$ is an integer, we introduce $q$ independent fields $\chi_i$

$$I = \int [D\phi] \int \prod_{1 \leq i \leq q} [D\chi_i] \, \mathrm{e}^{-\frac{1}{2}\phi^2} \mathrm{e}^{-\frac{1}{2v}\sum_i (\chi_i)^2} \mathrm{e}^{\frac{1}{v}\sum_i S[\chi_i + \phi]}$$

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It is convenient to trade $\chi_i$ for $\xi_i = \chi_i + \phi$ so that the integral over $\phi$ is Gaußian and can be performed

$$I = \int \prod_{1 \leq i \leq q} [D\xi_i] \, \mathrm{e}^{-\frac{1}{2v}\left\{\left(\sum_i (\xi_i)^2 - \frac{1}{v(1+u)}\left(\sum_i \xi_i\right)^2\right)\right\}} \times \mathrm{e}^{\frac{1}{v}\sum_i S[\xi_i]}.$$

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By expanding the integral over a multiplet of fields $\xi = (\xi_i)$ using Feynman diagrams, we generate the Tutte polynomials,

$$G_{\text{Tutte}}(u, v) = \frac{1}{u} \log\left\{\int [D\xi] \, \mathrm{e}^{-\frac{1}{2}\xi \cdot A^{-1}\xi} \, \mathrm{e}^{V(\xi)}\right\}$$

with a $q \times q$ propagator $A = v + M$, where $M$ is the $q \times q$ matrix whose entries are all equal to 1, and an interaction $V(\xi) = \frac{1}{v}\sum_i S(\xi_i)$.

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- sum over all the indices.

The evaluation of a graph $\gamma$ is proportional to the $q$-state **Potts model** partition function on $\gamma$,

$$Z(\beta, J, \gamma) = \sum_{\sigma} \mathrm{e}^{-\beta H(\sigma)}$$

where the sum runs over all states and $\beta$ is such that $v = \mathrm{e}^{-\beta J} - 1$. A state $\sigma$ is a assigment of spin in $q$ element set to each vertex of the graph and the Hamiltonian is

$$H(\sigma) = -\#\left\{\text{edges joining identical spins}\right\}$$

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We recover Postnikov's formula by evaluating $\frac{\partial S}{\partial \phi}$ at $s = 1$ and $\phi = 0$.

# Feynman graphs and their Symanzik polynomials

In **quantum field theory**, a Feynman diagram $\gamma$ with $n$ edges can be evaluated, in dimension $D$ as

$$\int \frac{d^n\alpha}{(U_\gamma(\alpha))^{\frac{D}{2}}}\, \mathrm{e}^{-\frac{V_\gamma(\alpha,p)}{U_\gamma(\alpha)}}$$

where $U_\gamma(\alpha)$ is the (first) **Symanzik polynomial**

$$U_\gamma(\alpha) = \sum_{\substack{t \\ \text{spanning trees}}} \prod_{i \notin t} \alpha_i$$

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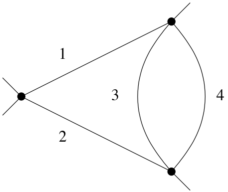


yields $U_\gamma(\alpha) = \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4$.

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since, in the Lie algebra of Feynman diagrams,

$$\frac{1}{n!} \underbrace{\Big[\delta_{\text{tree}}, \big[\cdots[\delta_{\text{tree}}, \delta_{\text{loop}}]\cdots\big]\Big]}_{n\,\text{iterations}} = \delta_{n\,\text{loop}}$$

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$U$  $(\alpha) = \frac{1}{2}\big\{(\alpha_1+\alpha_2+\alpha_3)\alpha_4 + (\alpha_1+\alpha_2+\alpha_4)\alpha_3 + (\alpha_1+\alpha_2)(\alpha_3+\alpha_4)\big\}$

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| rooted trees | Feynman diagrams |
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| non linear analysis | perturbative path integrals |
| fixed point equation | renormalization group equation |
| Powers of non linear operators $X^t(x)$ | background field technique $A^\gamma(S)$ |
| $x' = (\mathrm{id} - X)(x)$ <br> $= \sum_t \frac{X^t}{\mathrm{S}_t}$ | $S'[\phi] = \log \int [D\chi] \mathrm{e}^{-\frac{1}{2}\chi\cdot A\cdot\chi + S[\phi+\chi]}$ <br> $= \sum_\gamma \frac{A^\gamma(S)}{\mathrm{S}_\gamma}[\phi]$ |
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# Conclusion

Common framework for perturbative resolution of **non linear equations** and **effective actions** based on Hopf algebras of **rooted trees** and **Feynman diagrams**.

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Derivation of **combinatorial identities** (hook length formula, properties of the tutte polynomial, ...) inspired by **effective action computations**.