Pre-Lie algebras, rooted trees and posets

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Definition 1 *A pre-Lie algebra is a vector space W with a map $\curvearrowleft: W \otimes W \to W$ such that*

$$\begin{aligned}&(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z)\\ =&(x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y).\end{aligned}$$

Example 2 *All associative algebras are also pre-Lie algebras. Both sides vanish !*

Example 3 *The vector space of polynomial vector fields on affine space $\mathbb{A}^n$.*

$$\begin{aligned}\sum_i P(x)\partial_{x_i} \curvearrowleft \sum_j Q(x)\partial_{x_j}\\ = \sum_i \sum_j Q(x) \left(\partial_{x_j} P(x)\right) \partial_{x_i}. \quad (1)\end{aligned}$$

This is half the usual bracket of vector fields.

Indeed, in a pre-Lie algebra, the bracket

$$[x, y] = x \curvearrowleft y - y \curvearrowleft x, \qquad (2)$$

defines a Lie algebra, whence the name.

Example 4 *Every vertex algebra is also a graded pre-Lie algebra. The pre-Lie product is the normal ordered product of two fields, usually denoted by*

$$: a(z)b(z) : \tag{3}$$

Example 5 *Every operad gives rise to a pre-Lie algebra.*

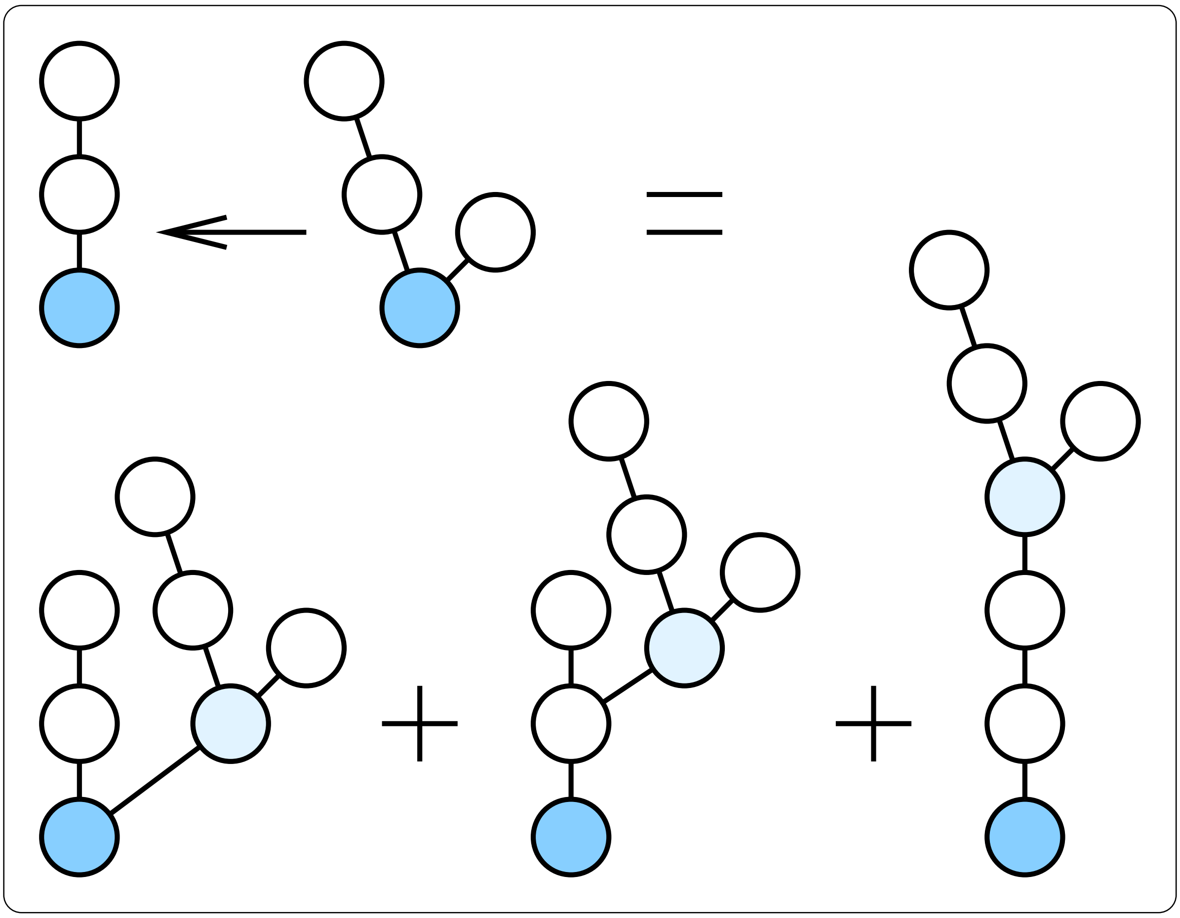
Example 6 *One can make pre-Lie algebras from many sorts of graphs by insertion at all possible places. For example, on the vector space spanned by trivalents graphs with three legs, define*

$$\Gamma_1 \curvearrowleft \Gamma_2 = \sum_v \sum_\sigma \Gamma_1 \circ_v^\sigma \Gamma_2, \tag{4}$$

where v runs over the set of vertices of Γ_1, σ over the set of bijections between the legs of Γ_2 and the edges at v of Γ_1 and $\circ_v^\sigma$ is the grafting at place v according to σ.

In this way, one gets many Lie algebras of graphs and Hopf algebras (their enveloping algebras). This is essentially what is used by Connes and Kreimer. Here is the reason why they also use rooted trees.

Theorem 7 (CL) *The free pre-Lie algebra on a single generator has a basis indexed by rooted trees. The pre-Lie product is given by the sum over all possible graftings.*



Corollary 8 *For a given polynomial vector field P, there exists a unique morphism from the free pre-Lie algebra on one generator O to the pre-Lie algebra of polynomial vector fields which maps O to P. To any rooted tree T, one can associate in this way a vector field T_P.*

Example 9 *Consider the following vector field (not polynomial, but analytic):*

$$V = \text{exp}(x)\partial_x. \tag{5}$$

Then for any rooted tree T, one has

$$T_V = |T| \,\text{exp}(x)\partial_x, \tag{6}$$

where $|T|$ is the number of vertices of T.

Example 10 *Find what is T_V for $V = x\partial_x$.*

Invariant bilinear forms on pre-Lie algebras

Let $(V, \curvearrowleft)$ be a pre-Lie algebra. An anti-symmetric bilinear form $\langle\, ,\, \rangle$ on V is called invariant if the following equations hold.

$$
\begin{aligned}
\langle x \curvearrowleft y\,,\, z\rangle &= -\langle x \curvearrowleft z\,,\, y\rangle,\\
\langle x \curvearrowleft y\,,\, z\rangle &= -\langle y \curvearrowleft z\,,\, x\rangle + \langle z \curvearrowleft y\,,\, x\rangle.
\end{aligned}
$$

The radical of an invariant form $\langle\, ,\, \rangle$ is a two sided pre-Lie ideal.

Question 11 *Find interesting examples !*

The free pre-Lie algebras have many invariant forms.

More generally, it would be nice to have interesting examples of finite-dimensional pre-Lie algebras.

Some modules over the symmetric groups

Consider the free pre-Lie algebra on the set $\{x_1, \ldots, x_n\}$. Let $\mathrm{PreLie}(n)$ be the subspace spanned by products where each x_i appear exactly once. There is an action of the symmetric group $\mathfrak{S}_n$ on $\mathrm{PreLie}(n)$ by changing the labels.

By the previous theorem describing free pre-Lie algebras,

Theorem 12 *The $\mathfrak{S}_n$-module* $\mathrm{PreLie}(n)$ *is given by the action of $\mathfrak{S}_n$ on rooted trees labeled by the set $[n] = \{1, \ldots, n\}$. It has dimension n^{n-1}.*

But the notion of invariant bilinear form gives more : there is a hidden action !

Theorem 13 *There is an action of the symmetric group $\mathfrak{S}_{n+1}$ on* $\mathrm{PreLie}(n)$ *which extends the action of $\mathfrak{S}_n$.*

The poset of pointed partitions

This poset was introduced recently by B. Vallette, motivated by Koszul duality of operads.

A pointed partition of $[n]$ is a set-partition π of $\{1, \ldots, n\}$ together with the choice of one element in each part of π. For example:

$$\{\overline{\mathbf{4}}, 7, 2\}\,\{\overline{\mathbf{5}}, 1\}\,\{\overline{\mathbf{6}}, 3\}. \tag{7}$$

The partial order is defined as follows $\pi \leq \lambda$ if the partitions are related by refinement and the set of pointed elements of the coarser λ is contained in the set of pointed elements of the finer π. For example :

$$\{\overline{\mathbf{4}}\}\,\{\overline{\mathbf{1}}\}\,\{\overline{\mathbf{3}}, 2\} \leq \{\overline{\mathbf{4}}, 1\}\,\{\overline{\mathbf{3}}, 2\}. \tag{8}$$

There is one unique minimal element and n maximal elements. This poset is not a lattice.

Then one has

Theorem 14 (Vallette) *The top homology of the poset of pointed partitions of $[n]$ is isomorphic to the $\mathfrak{S}_n$-module* $\mathrm{PreLie}(n)$ *(up to tensor product by the sign module).*

This was obtained using some known properties of the pre-Lie operad.

Theorem 15 (CV) *The poset of pointed partitions of $[n]$ is shellable and Cohen-Macaulay. Its characteristic polynomial is $(x-n)^{n-1}$.*

From this follows that the homology is concentrated in maximal degree.

There is an interesting related poset.

Theorem 16 (CV) *The bounded poset obtained by adjoining a maximal element to the poset of pointed partitions of $[n]$ is also shellable and Cohen-Macaulay. Its Möbius number is $(n-1)^{n-1}$ (up to sign).*

Hence, there is an interesting action of $\mathfrak{S}_n$ on its top homology.

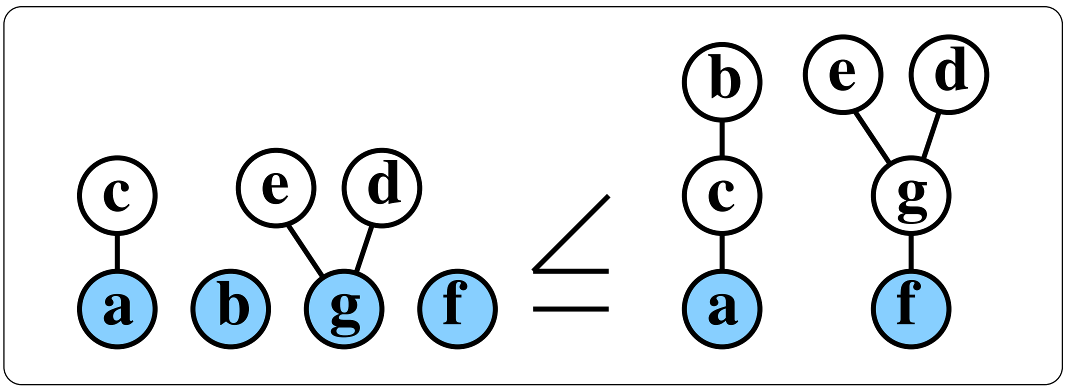
The poset of rooted forests.

This poset was introduced by J. Pitman, in his study of coalescent random forests.

The underlying set is the set of forests of rooted trees on $[n]$: graphs with no cycles on the vertex set $[n]$ with a chosen element called the root in each connected component.

To define the partial order, one has to see each edge as an arrow directed towards the root. Then $F_1 \leq F_2$ if the forest F_1 can be obtained from F_2 by removal of some edges. When removing an edge, the new roots are given by the vertices with no out-coming arrows.

Here is an example:



There is a unique minimal element. There are n^{n-1} maximal elements which are connected forests *i.e.* rooted trees.

There may be a nice relation to the poset of pointed partitions :

Question 17 *Is the characteristic polynomial* $(x-n)^{n-1}$ *?*

Question 18 *Is this poset shellable ?*

Question 19 *Is the action on top homology the same as for the poset of pointed partitions ?*

There is an obvious forgetful morphism of posets from rooted forests to pointed partitions.

Question 20 *Does this morphism give a homotopy equivalence ?*

Question 21 *Same questions about the poset with a top element added ?*

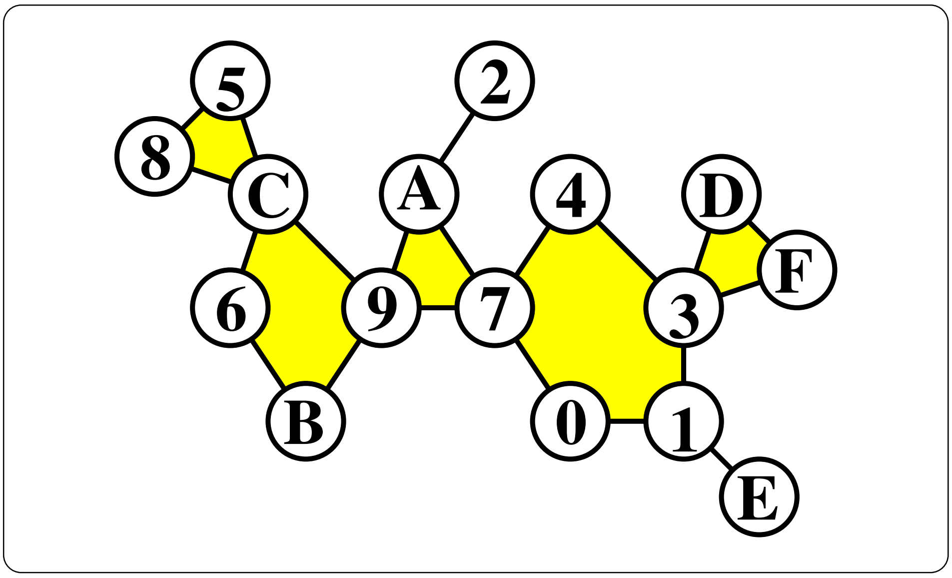
The poset of hypertrees

This name was given by J. McCammond and J. Meier, in their study of automorphism groups of free groups. The poset was defined before using the equivalent language of white-labelled bipartite trees.

An hypergraph on the finite set $[n]$ is a set of subsets of $[n]$ of cardinality at least 2. These subsets are called the edges. There is a clear notion of path in an hypergraph. So one can define connected hypergraphs and cycles in hypergraphs.on 16 vertices,

An hypertree on the finite set $[n]$ is a connected hypergraph on $[n]$ with no cycle. This implies that distinct edges share at most one vertex.

Here is an example of hypertree on $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$.



The partial order is defined as follows : one has $H_1 \leq H_2$ if each edge of H_1 is obtained by taking the union of some edges of H_2.

There is a unique minimal element, which has only one edge. There are n^{n-2} maximal elements, which are trees.

Theorem 22 (McCammond-Meier) *The poset obtained by adding a maximal element is shellable and Cohen-Macaulay. Its Möbius number is* $(n-1)^{n-2}$ *up to sign.*

Question 23 *Is there an isomorphism between the $\mathfrak{S}_n$ action on the top homology module of the augmented hypertree poset and the $\mathfrak{S}_n$ hidden action on* $\mathrm{PreLie}(n-1)$ *?*

The pre-Lie graph complex

Recall that, after Kontsevitch, one can define a graph complex for each cyclic operad.

But the pre-Lie operad is anticyclic, *i.e.* its suspension is a cyclic operad. Hence there exists a pre-Lie graph complex.

Question 24 *What does its homology compute ?*

Recall for example that the homology of the graph complex for the Lie operad is related to Vassiliev knot invariants.