Pre-Lie algebras, rooted trees and related algebraic structures

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Definition 1 *A pre-Lie algebra is a vector space W with a map $\curvearrowleft : W \otimes W \to W$ such that*

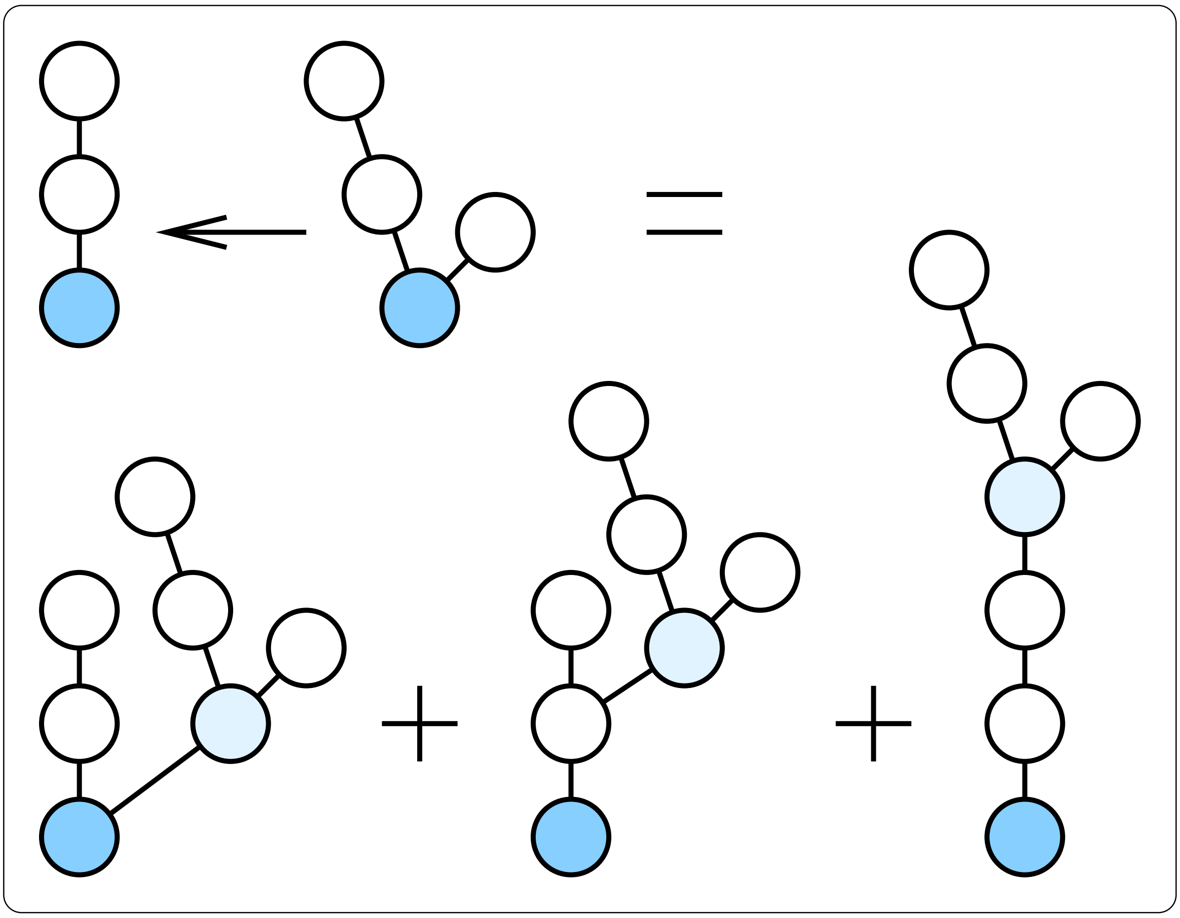
$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y). \quad (1)$$

Example 2 *All associative algebras are also pre-Lie algebras.*

Example 3 *The vector space of polynomial vector fields on affine space $\mathbb{A}^n$.*

$$\sum_i P(x)\partial_{x_i} \curvearrowleft \sum_j Q(x)\partial_{x_j} = \sum_i \sum_j Q(x) \left(\partial_{x_j} P(x)\right) \partial_{x_i}. \quad (2)$$

Theorem 4 (CL) *The free pre-Lie algebra on a single generator has a basis indexed by rooted trees. The pre-Lie product is given by the sum over all possible graftings.*



Corollary 5 *For a given polynomial vector field P, there exists a unique morphism from the free pre-Lie algebra on one generator O to the pre-Lie algebra of polynomial vector fields which maps O to P. To any rooted tree T, one can associate in this way a vector field T_P.*

Example 6 *Consider the following vector field (not polynomial, but analytic):*

$$V = \exp(x)\partial_x. \tag{3}$$

Then for any rooted tree T, one has

$$T_V = |T| \exp(x)\partial_x, \tag{4}$$

where $|T|$ is the number of vertices of T.

Example 7 *Find what is T_V for $V = x\partial_x$.*

Definition 8 *The* PreLie *operad is the operad describing pre-Lie algebras.*

Recall that an operad $\mathcal{P}$ is given by a collection of modules $\mathcal{P}(n)$ over the symmetric groups $\mathfrak{S}_n$ with composition maps

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1), \qquad (5)$$

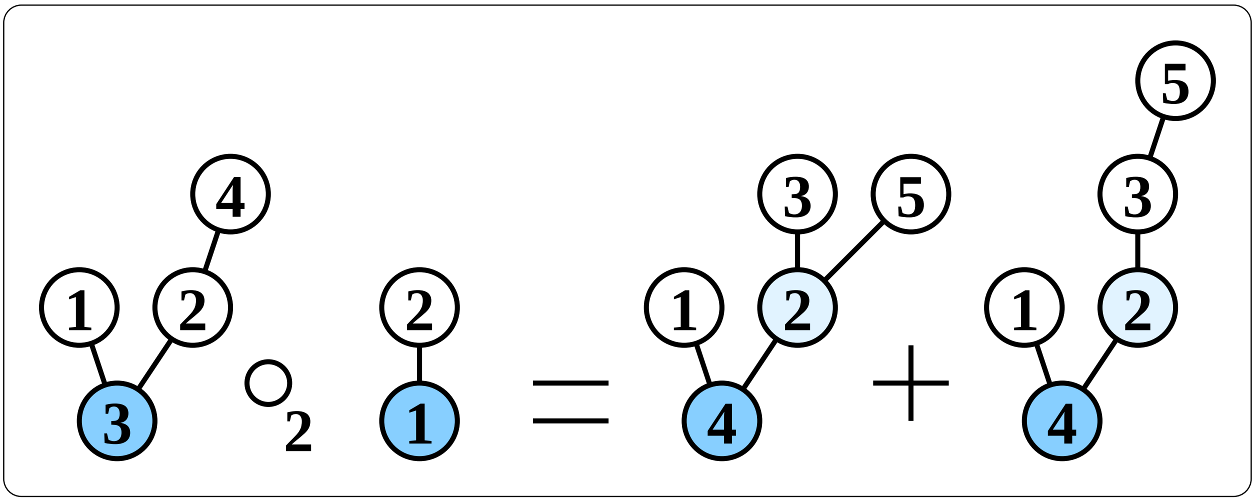
satisfying natural axioms. The standard example is given by

$$\mathcal{P}(n) = \mathrm{hom}(W^{\otimes n}, W) \qquad (6)$$

for some fixed vector space W, together with composition of multi-linear operations at position i.

One can reformulate and enhance the description of the free pre-Lie algebras as a description of the PreLie operad.

Theorem 9 (CL) *The vector space* $\mathrm{PreLie}(n)$ *has a basis indexed by the set of rooted trees with vertices in bijection with* $\{1, \ldots, n\}$ *(labelled rooted trees). The action of* $\mathfrak{S}_n$ *is by changing the decoration. The composition* $T \circ_i T'$ *of a tree* T' *at place* i *of a tree* T *is a sum over the set of maps from incoming edges at* i *to vertices of* T'.



For any collection $\mathcal{P}(n)$ of $\mathfrak{S}_n$-modules, one defines an analytic functor which maps a vector space W to

$$\mathcal{P}(W) = \bigoplus_{n \geq 1} W^{\otimes n} \otimes_{\mathfrak{S}_n} \mathcal{P}(n). \tag{7}$$

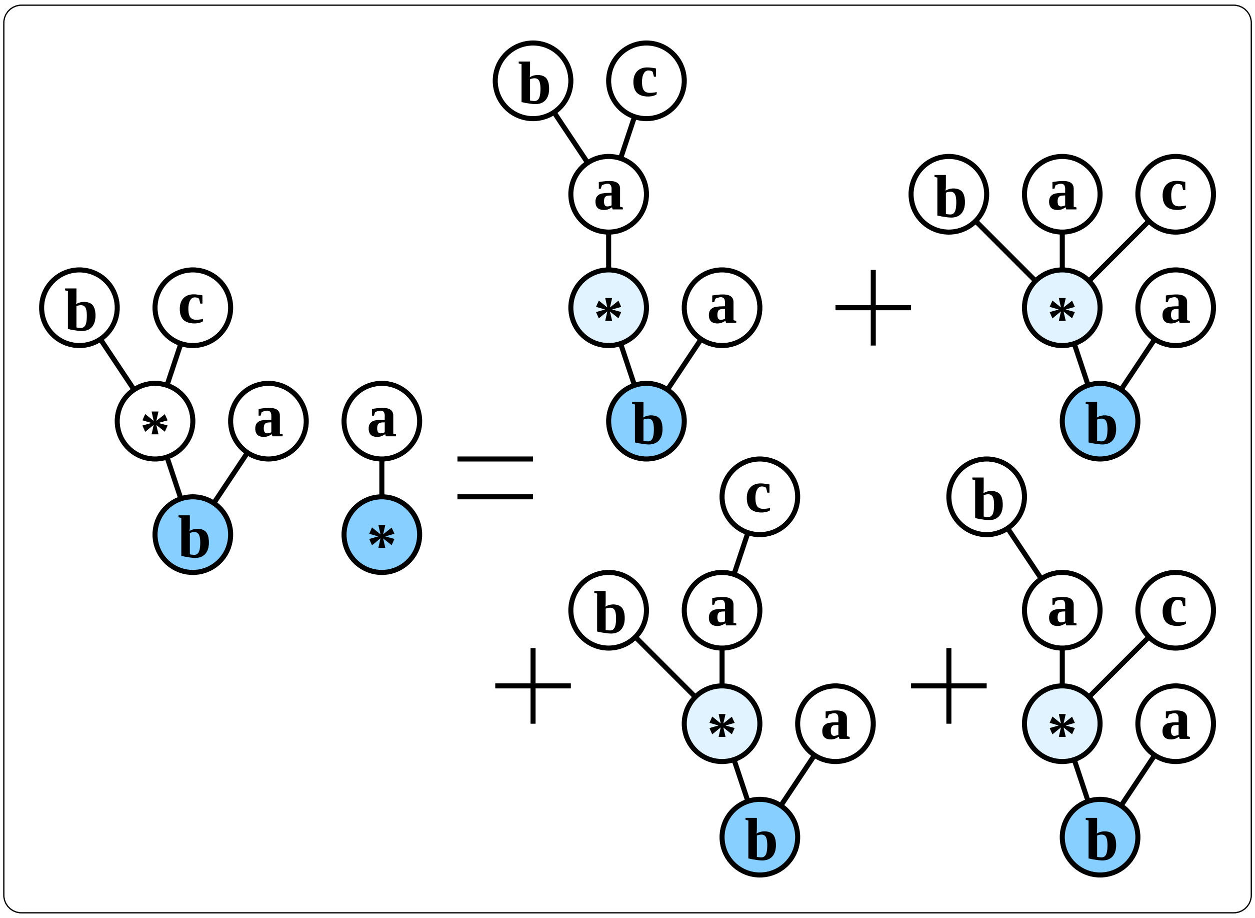
Recall that one can define the derived functor $\mathcal{P}'$ of an analytic functor $\mathcal{P}$ by the formula

$$\mathcal{P}'(W) = \mathcal{P}(W \oplus \mathbb{K}\{*\}). \tag{8}$$

Equivalently, the $\mathfrak{S}_n$-module $\mathcal{P}'(n)$ is the restriction of the action of $\mathfrak{S}_{n+1}$ on $\mathcal{P}(n+1)$ to the subgroup fixing $n+1$.

Theorem 10 *If $\mathcal{P}$ is an operad, then, for any vector space W, $\mathcal{P}'(W)$ has a natural structure of associative algebra. The product is given by composition $\circ_*$ at the distinguished place $*$.*

Example 11 *Let us consider the case of the* PreLie *operad and a vector space* $W = \mathbb{K}\{a, b, c\}$. *Then* $\mathrm{PreLie}'(W)$ *has a basis indexed by rooted trees with a map from vertices to the set* $\{a, b, c, *\}$ *such that* $*$ *is the image of exactly one vertex, called the distinguished vertex (marked decorated rooted trees).*



Theorem 12 *The subspace* $\mathcal{I}(W)$ *spanned by trees where* $*$ *is not the root is a two-sided ideal of* $\mathrm{PreLie}'(W)$*.*

The quotient algebra has another description. Recall that the bracket

$$[x, y] = x \curvearrowleft y - y \curvearrowleft x, \qquad (9)$$

in a pre-Lie algebra defines a Lie algebra.

Theorem 13 *The quotient of* $\mathrm{PreLie}'(W)$ *by* $\mathcal{I}(W)$ *is isomorphic as an associative algebra to the universal enveloping algebra of the Lie algebra associated to the free pre-Lie algebra on* W*.*

$$\mathrm{PreLie}'(W)/\mathcal{I}(W) \simeq U(\mathrm{PreLie}(W)_{\mathrm{Lie}}). \qquad (10)$$

N.B. This enveloping algebra is the dual of the Hopf algebra of rooted trees, see Butcher, Dür, Grossman & Larsson, Connes & Kreimer, Hoffman, Foissy etc..

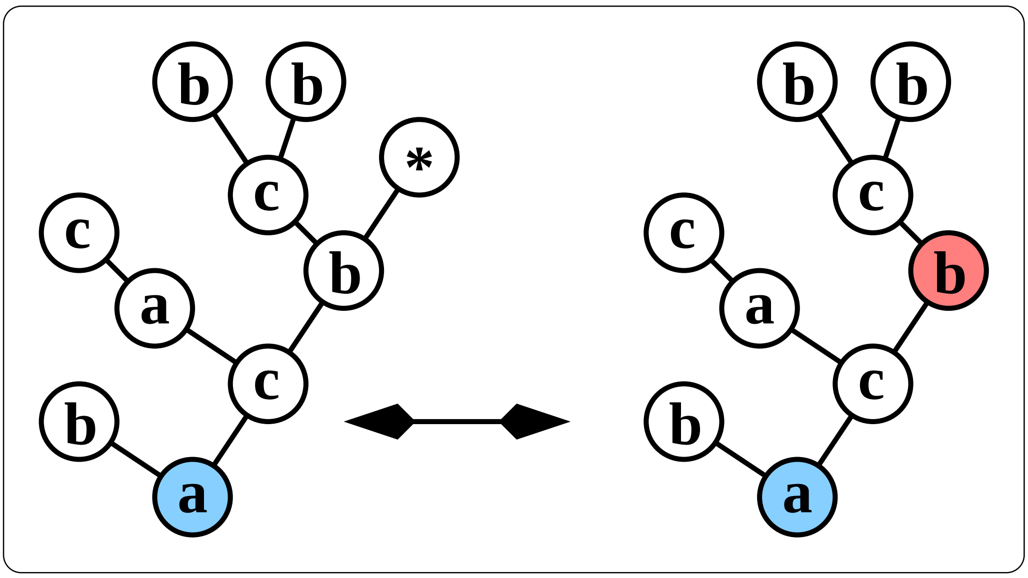
Definition 14 *A leaf is a vertex without incoming edges.*

Theorem 15 *The subspace $\mathcal{Q}(W)$ spanned by trees where $*$ is a leaf is a sub-algebra of* $\mathrm{PreLie}'(W)$.

Definition 16 *A vertebrate is a rooted tree with a distinguished vertex called the tail.*

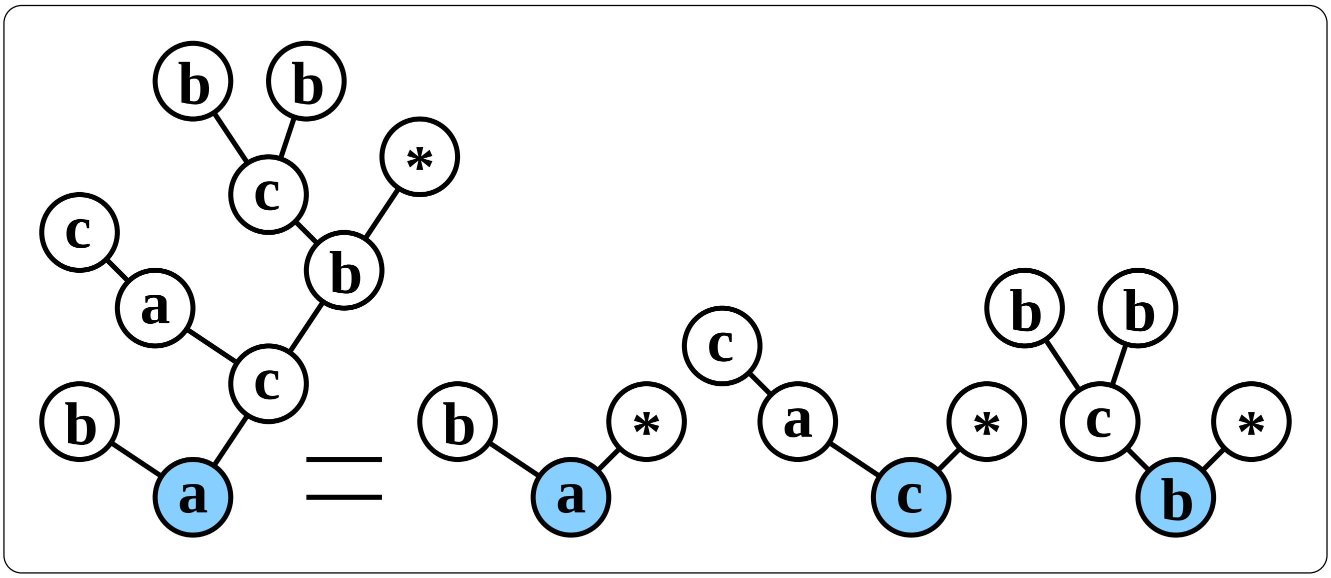
N.B. The tail can be the same as the root.

A rooted tree where $*$ is a leaf can be considered as a vertebrate: the distinguished vertex is removed and the tail is the vertex to which is was grafted.



Theorem 17 *The algebra $\mathcal{Q}(W)$ is the free associative algebra on the subspace spanned by rooted trees where $*$ is a leaf attached to the root.*

When considered as vertebrates, the generators are the vertebrates where the tail is the root. They can be identified with rooted trees.



Theorem 18 (Foissy) *The universal enveloping algebra* $U(\mathsf{PreLie}(W)_{\mathsf{Lie}})$ *is a free associative algebra.*

No explicit description of a subspace $\mathcal{M}(W)$ of generators is known.

Theorem 19 *The algebra* $\mathsf{PreLie}'(W)$ *is a free associative algebra on the space of generators* $\mathcal{M}(W) \oplus W$*, where the generators in the part* W *are the trees with two vertices whose vertex marked by* $*$ *is not the root.*

Generating functions

To each collection $\mathcal{P}(n)$ of $\mathfrak{S}_n$-modules, one associates a generating function:

$$P = \sum_{n \geq 0} \dim \mathcal{P}(n)\, x^n / n!. \tag{11}$$

For PreLie, one gets

$$\mathsf{PL} = \sum_{n \geq 1} n^{n-1}\, x^n / n!, \tag{12}$$

closely related to the Lambert W-function.

Many algebraic theorems on free pre-Lie algebras imply analytic properties of the function PL. For example:

Theorem 20 (CL) *The free pre-Lie algebra* $\mathrm{PreLie}(W)$ *is a free module over the universal enveloping algebra* $U(\mathrm{PreLie}(W)_{\mathrm{Lie}})$ *over the generators* W*. Hence one has*

$$\mathsf{PL} = x \exp(\mathsf{PL}). \tag{13}$$

From now on, replace the category of vector spaces by the category of chains complexes with vanishing differentials.

Definition 21 *A* Λ*-algebra is a chain complex* W *with a map* $\curvearrowleft : W \otimes W \to W$ *of degree* 0 *and a map* $\langle\, , \rangle : W \otimes W \to W$ *of degree* 1 *such that* $\curvearrowleft$ *is a pre-Lie product,* $\langle\, , \rangle$ *is a Lie bracket of degree* 1 *and the following relation holds*

$$\pm\langle x \curvearrowleft y\,,\, z\rangle \pm \langle z \curvearrowleft y\,,\, x\rangle = \langle z\,,\, x\rangle \curvearrowleft y, \quad (14)$$

where appropriate signs have to be inserted according to the Koszul sign rules.

Many good properties of pre-Lie algebras should generalize to Λ-algebras.

Conjecture 22 *The generating series for the analytic functor* Λ *is given by*

$$\Lambda = \sum_{n \geq 1} \left(\prod_{k=1}^{n-1} (n - k\, t) \right) x^n / n!, \qquad (15)$$

This would follow from the Koszul property for the operad Λ.

These dimensions are known to be upper bounds. Note that one recovers PL when $t = 0$.

Conjecture 23 *The free* Λ*-algebra* $\Lambda(W)$ *is free as a Lie algebra and its enveloping algebra is free as an associative algebra.*

Conjecture 24 *The associative algebra* $\Lambda'(W)$ *is a free associative algebra.*

Conjecture 25 *There exists a quotient map of associative algebras*

$$\Lambda'(W) \longrightarrow U(\Lambda(W)_{\mathrm{Lie}}). \qquad (16)$$