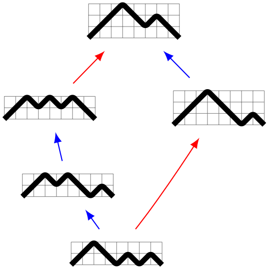
Dexter semi-lattices and Hochschild polytopes

Frédéric Chapoton

CNRS & Université de Strasbourg

Janvier 2019



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→ (C) binary trees under left-arm rotation order (Pallo),
→ (D) Dyck paths for inclusion, 
→ (E) Dyck paths and Tamari sliding moves (equivalent to (B)),
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and still others by restriction from the symmetric groups.

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Today, introduce yet another one :
→ Dyck paths and dexter sliding moves.

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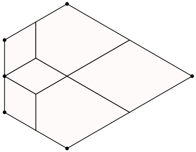
in dimension 2: 

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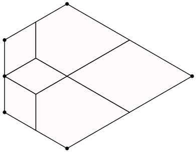
in dimension 2: 

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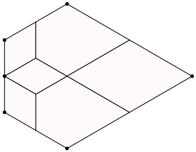
in dimension 2: 

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Note the natural (visual) partition into cells

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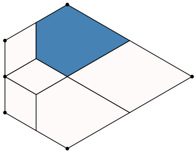

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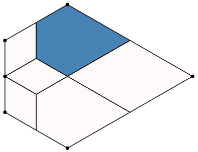

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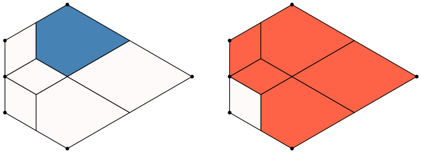

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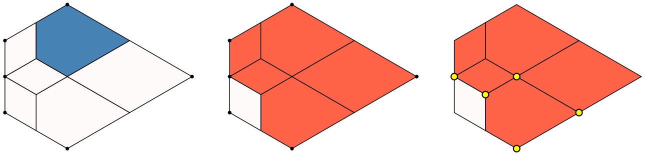

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This gives Catalan-many cells (among many more cells A0139).
$\rightarrow$ induced partial order on the **bottom elements** of these cells

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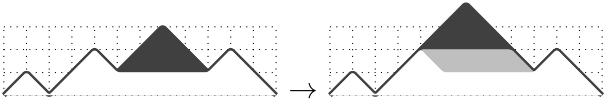

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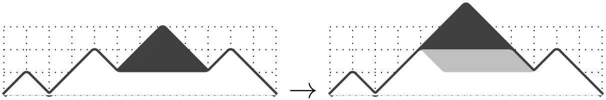

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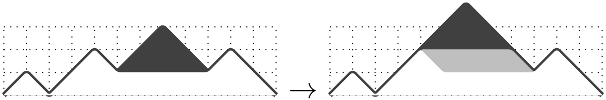

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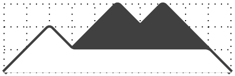
→ Every dexter sliding move is like a sequence of Tamari sliding moves, so something like a **shortcut** in the Tamari lattice

More examples of sliding moves



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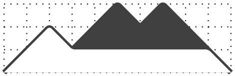

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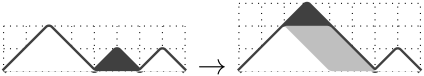
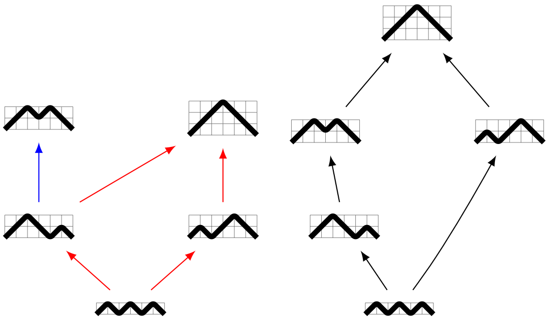


one single dexter sliding move.
This would be two consecutive moves in the Tamari lattice.

Comparison between Tamari and Dexter



Dexter on the left and Tamari on the right

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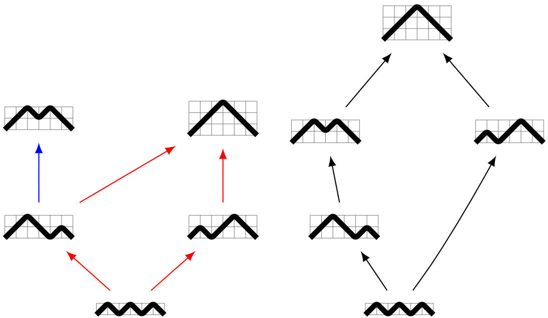

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Tamari has strictly more relations than Dexter.

Picture of the next full Dexter poset

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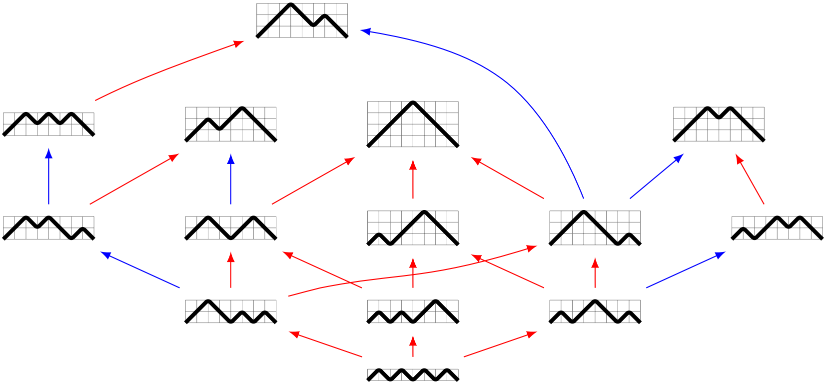

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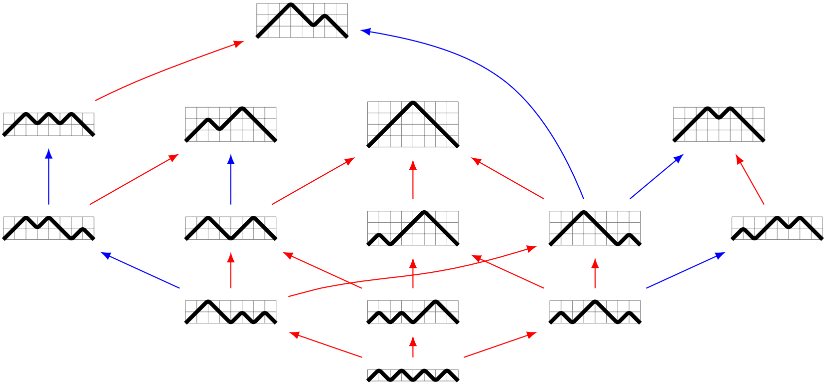

Can you see the hidden pentagon ?

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Corollary

Every interval in $\mathscr{D}_n$ is a lattice.

Enumeration of intervals (closed formula)

Theorem

The number of intervals in the poset $\mathscr{D}_n$ is 1 for $n = 0$ and

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→ (A) numbers of **rooted bicubic planar maps** on $2n$ vertices
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Bijection between (A) and (B) is classical (Tutte).
Rognerud has given a simple bijection between (C) and (D).

About the proof (bijections and formulas)

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$\rightarrow$ the known algebraic equation for the sequence A257

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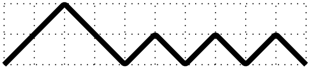
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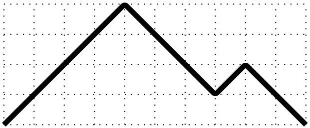

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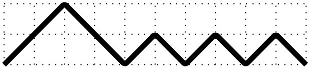
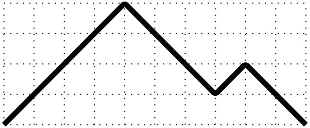
 and 

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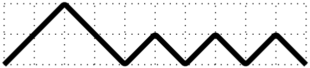
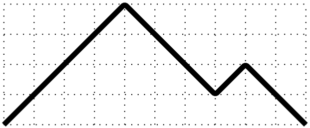
 and 

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Call F_n the set of elements in this interval (with n little peaks).
This is a lattice (because all intervals are).

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Not graded, hence not distributive. Maybe a trim lattice ?

Bonus tracks (conjectures, remarks, speculations)

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With no colors: same h-vector as the associahedra (Narayana numbers)
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Questions ?