# ON A GROUP OF SERIES INDEXED BY ROOTED TREES

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Abstract. This paper introduces a group of generalized power series associated to any augmented operad, then focuses on the case of the PreLie operad, where the series are indexed by rooted trees. The solution of flow equations using the pre-Lie structure on vector fields on an affine space gives rise to an interesting element of this group.

## 0. Introduction

This paper is composed of three main ingredients: the first is the definition of a group of power series $\mathsf{G}_{\mathcal{P}}$ for any augmented operad $\mathcal{P}$. A related Lie algebra has been introduced before, from another point of view, by Kapranov and Manin [5].

The rest of the paper focuses on the case of the PreLie operad [4]. The group $\mathsf{G}_{\mathrm{PreLie}}$ is a group of series indexed by rooted trees. The second component of the paper is given by two quotient operads of PreLie corresponding respectively to linear trees and corollas. The operad on linear trees is just the associative operad As, whereas the operad Mu on corollas seems to be new. By functoriality, one gets group maps $\mathsf{G}_{\mathrm{PreLie}} \to \mathsf{G}_{\mathrm{As}}$ and $\mathsf{G}_{\mathrm{PreLie}} \to \mathsf{G}_{\mathrm{Mu}}$. Whereas $\mathsf{G}_{\mathrm{As}}$ is isomorphic to the group of formal power series for the composition product, $\mathsf{G}_{\mathrm{Mu}}$ is related to the group of formal power series for the pointwise multiplication product.

The final ingredient comes from flow equations for vector fields on the affine space $\mathsf{A}^n$. We recall the formula which gives the Taylor expansion of the solution of a flow equation using the pre-Lie structure on the vector space of vector fields on $\mathsf{A}^n$. We have not traced the precise origin of this formula, although it is certainly not new, see [1] and references therein. This formula can be interpreted as a distinguished element of the group $\mathsf{G}_{\mathrm{PreLie}}$, which we choose to denote by $\exp^*$, for its image in $\mathsf{G}_{\mathrm{As}}$ is $\exp -1$, where exp is the usual exponential function.

## 1. A group associated to an augmented operad

Let $\mathcal{P}$ be an operad in the category $\mathrm{Vect}_{\mathbb{Q}}$ of vector spaces over $\mathbb{Q}$ and assume that $\mathcal{P}(0) = \{0\}$ and that $\mathcal{P}(1) = \mathbb{Q}e$ where $e$ is the unit of $\mathcal{P}$. Such an operad is called augmented.

Let $\mathsf{F}\mathcal{P} = \oplus_n \mathcal{P}(n)_{\mathfrak{S}_n}$ be the direct sum of the coinvariant spaces, which can be identified with the underlying vector space of the free $\mathcal{P}$-algebra on a generator $v$, and $\widehat{\mathcal{P}} = \prod_n \mathcal{P}(n)_{\mathfrak{S}_n}$ be its completion.

Let $x = \sum_m x_m$, $y = \sum_n y_n$ be two elements of $\widehat{\mathcal{P}}$. Choose any representatives $\bar{x}_m$ of $x_m$ (resp. $\bar{y}_n$ of $y_n$) in the operad $\mathcal{P}$. Then one can check that the following

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formula defines a product on $\widehat{\mathcal{P}}$.

$$x \times y = \sum_{m \geq 1} \sum_{n_1,\dots,n_m \geq 1} \langle \gamma(\bar{x}_m, \bar{y}_{n_1}, \dots, \bar{y}_{n_m}) \rangle, \tag{1}$$

where $\langle\,\rangle$ is the quotient map to the coinvariants and $\gamma$ is the composition map of the operad $\mathcal{P}$.

**Proposition 1.** *The product $\times$ defines the structure of an associative monoid on the vector space $\widehat{\mathcal{P}}$. Furthermore, this product is $\mathbb{Q}$-linear on its left argument.*

*Proof.* Let us first prove the associativity. On the one hand, one has

$$\begin{aligned}(x \times y) \times z &= \sum_{m} \sum_{p_1,\dots,p_m} \langle \gamma(\overline{(x \times y)}_m, \bar{z}_{p_1}, \dots, \bar{z}_{p_m}) \rangle \\ &= \sum_{m} \sum_{n_1,\dots,n_m} \sum_{p_1,\dots,p_{n_1+\dots+n_m}} \langle \gamma(\gamma(\bar{x}_m, \bar{y}_{n_1}, \dots, \bar{y}_{n_m}), \bar{z}_{p_1}, \dots, \bar{z}_{p_{n_1+\dots+n_m}}) \rangle.\end{aligned} \tag{2}$$

On the other hand, one has

$$\begin{aligned}x \times (y \times z) &= \sum_{m} \sum_{n_1,\dots,n_m} \langle \gamma(\bar{x}_m, \overline{(y \times z)}_{n_1}, \dots, \overline{(y \times z)}_{n_m}) \rangle \\ &= \sum_{m} \sum_{n_1,\dots,n_m} \sum_{(q_{i,j})} \langle \gamma(\bar{x}_m, \gamma(\bar{y}_{n_1}, \bar{z}_{q_{1,1}}, \dots, \bar{z}_{q_{1,n_1}}), \dots, \gamma(\bar{y}_{n_m}, \bar{z}_{q_{m,1}}, \dots, \bar{z}_{q_{m,n_m}})) \rangle.\end{aligned} \tag{3}$$

Using then the "associativity" of the operad, one gets the associativity of $\times$. It is easy to check that the image of the unit $e$ of the operad $\mathcal{P}$ is a two-sided unit for the $\times$ product. The left $\mathbb{Q}$-linearity is clear from the formula (1). $\square$

**Proposition 2.** *An element $y$ of $\widehat{\mathcal{P}}$ is invertible for $\times$ if and only if the first component $y_1$ of $y$ is non-zero.*

*Proof.* The direct implication is trivial. The reverse one is proved by a very standard recursive argument. $\square$

Let us call $\mathsf{G}_{\mathcal{P}}$ the set of invertible elements of $\widehat{\mathcal{P}}$ for the $\times$ product.

**Proposition 3.** *$\mathsf{G}$ is a functor from the category of augmented operads to the category of groups.*

*Proof.* The functoriality follows from inspection of the definitions of $\widehat{\mathcal{P}}$ and $\times$. $\square$

In fact, one can see $\mathsf{G}_{\mathcal{P}}$ as the group of $\mathbb{Q}$-points of a pro-algebraic group. The Lie algebra of this pro-algebraic group is given by the usual linearization process on the tangent space $\widehat{\mathcal{P}}$, resulting in the formula

$$[x, y] = \sum_{m \geq 1} \sum_{n \geq 1} \langle \bar{x}_m \circ \bar{y}_n - \bar{y}_n \circ \bar{x}_m \rangle, \tag{4}$$

where

$$\bar{x}_m \circ \bar{y}_n = \sum_{i=1}^{m} \gamma(\bar{x}_m, \underbrace{e, \dots, e}_{i-1 \text{ units}}, \bar{y}_n, e, \dots, e). \tag{5}$$

The graded Lie algebra structure on $\mathsf{F}\mathcal{P}$ defined by the same formulas has already appeared in the work of Kapranov and Manin on the category of right modules over an operad [5, Th. 1.7.3]. They explained that it acts by polynomial vector fields on the underlying vector space of any $\mathcal{P}$-algebra. For the endomorphism operad $\mathrm{End}(L)$ of a vector space $L$, $\mathsf{G}_{\mathrm{End}(L)}$ is the group of formal diffeomorphisms of $L$ preserving the origin. So $\mathsf{G}_{\mathcal{P}}$ acts by formal diffeomorphisms on any $\mathcal{P}$-algebra $L$.

## 2. The flow of a vector field

Let $\mathsf{A}^n$ be the affine space of dimension $n$ over $\mathbb{R}$. It is well-know [3, 7] that there is a structure of pre-Lie algebra on the vector space $\mathsf{V}_n$ of smooth vector fields on $\mathsf{A}^n$ (see [4] for the notion of pre-Lie algebra). More precisely, let $x_1, \dots, x_n$ be coordinates on $\mathsf{A}^n$. Given two vector fields $F = \sum F_i \partial_i$ and $F' = \sum F'_j \partial_j$, their pre-Lie product $F \leftarrow F'$ is given by

$$F \leftarrow F' := \sum \sum F'_j (\partial_j F_i) \partial_i. \tag{6}$$

This does not depend on the choice of affine coordinates.

Let $F \in \mathsf{V}_n$ be a vector field. The flow equation of $F$ is the following equation for a smooth function $g : \mathsf{A}^1 \to \mathsf{A}^n$ of the variable $t$:

$$\begin{cases} \frac{dg}{dt} = F(g), \\ g(0) = g_0, \end{cases} \tag{7}$$

where $g_0$ is any chosen point in $\mathsf{A}^n$. Smoothness of $F$ ensures uniqueness of the solution.

One can give a formal Taylor development at $t = 0$ of the solution of (7) using the following construction in the pre-Lie algebra of vector fields (see [3, Prop. 4]). Let $F^{\leftarrow k}$ denote the $k^{th}$ right iterate of $F$ for the pre-Lie product, *i.e.*

$$F^{\leftarrow k} = \underbrace{((\cdots((F \leftarrow F) \leftarrow F) \cdots \leftarrow F) \leftarrow F)}_{k \text{ factors}}. \tag{8}$$

**Proposition 4.** *The solution of the flow equation (7) has the following formal Taylor expansion:*

$$g(t) = g_0 + \sum_{k \geq 1} F^{\leftarrow k}(g_0) \frac{t^k}{k!}. \tag{9}$$

The proof is given in [3].

In the special case where the vector field is linear, *i.e.* $F = \sum a_{i,j} x_i \partial_j$ with $a_{i,j} \in \mathbb{R}$, the formula (9) gives back the classical exponential formula:

$$g(t) = g_0 + (\exp(tA) - I) g_0 = \exp(tA) g_0. \tag{10}$$

where $A$ is the matrix $(a_{i,j})$.

One can define an element $\exp^*(v)$ of the completed free pre-Lie algebra on a generator $v$ over $\mathbb{Q}$ by

$$\exp^*(v) := \sum_{k \geq 1} \frac{v^{\leftarrow k}}{k!}. \tag{11}$$

**Definition 1.** *The image of* $\exp^*(v)$ *by the usual identification between the completed free pre-Lie algebra on* $\{v\}$ *and* $\widehat{\mathrm{PreLie}}$ *is an element of the group* $\mathsf{G}_{\mathrm{PreLie}}$*, which is denoted by* $\exp^*$*. Its inverse in this group is denoted by* $\log^*$*.*

The group $\mathsf{G}_{\mathrm{PreLie}}$ acts on the completed free pre-Lie algebra on $v$, and the action of $\exp^*$ on $v$ is $\exp^*(v)$.

If $F$ is a vector field on $\mathsf{A}^n$, then the difference between the flow at time 0 and the flow at time 1 can be considered as another vector field $G$ on $\mathsf{A}^n$. In fact, formula (9) says that it is formally given by $\exp^* F$. So the meaning of $\log^*$ is the reverse operation: knowing the displacement $G$ between time 0 and time 1, $\log^* G$ formally recovers the vector field $F$.

## 3. Linear trees and composition

As shown in [4], the PreLie operad can be described in terms of labeled rooted trees. By convention, edges are oriented towards the root. Let us call linear the trees that do no branch, that is to say all vertices have at most one incoming edge.

We recall here briefly (see [4] for more details) the definition of the composition of two labeled rooted trees $T$ and $S$ on the vertex sets $I$ and $J$ respectively. Let $i \in I$; the composition of $S$ at the vertex $i$ of $T$ is given by

$$T \circ_i S = \sum_f T \circ_i^f S, \tag{12}$$

where the sum runs over all maps $f$ from the set of incoming edges of the vertex $i$ of $T$ to the set of vertices of $S$, and $T \circ_i^f S$ can be described as follows: replace the vertex $i$ by the tree $S$, grafting back the subtrees of $T$ previously attached to $i$, according to the map $f$.

**Proposition 5.** *The subspace of* PreLie *spanned by non-linear labeled trees is an ideal. The quotient map $\phi$ coincides with the usual map from* PreLie *to the associative operad* As*.*

*Proof.* Using the description above of the composition map of the operad PreLie, it is clear that the composition of two labeled trees, at least one of which is non-linear, is again non-linear. The quotient operad, spanned by labeled linear trees, has dimension $n!$ in rank $n$. Its composition can be easily identified with the associative operad As. The quotient map is then checked on generators of PreLie to be the same as the usual map. □

By functoriality of the group construction, there is a map, still denoted by $\phi$, from $\mathsf{G}_{\mathrm{PreLie}}$ to $\mathsf{G}_{\mathrm{As}}$.

**Proposition 6.** *The group $\mathsf{G}_{\mathrm{As}}$ is isomorphic to the group of invertible formal power series in $x\mathbb{Q}[[x]]$ for the composition product.*

*Proof.* It is more convenient here to work at the monoid level with $\widehat{\mathrm{As}}$ and $x\mathbb{Q}[[x]]$. The vector space $\mathrm{As}(n)_{\mathfrak{S}_n}$ is one dimensional for all $n$, with a basis given by the image by $\phi$ of the linear tree with $n$ nodes. Let us denote this basis element by $\theta_n$. By left linearity of both monoids, it is sufficient to check the product rule for $\theta_m$ and $f = \sum_{n\geq 1} f_n \theta_n$. One finds that

$$\theta_m \times f = \sum_{n_1,\dots,n_m \geq 1} f_{n_1} \dots f_{n_m} \theta_{n_1+\dots+n_m}, \tag{13}$$

which proves that the linear map defined by $x^n \mapsto \theta_n$ is an isomorphism between the monoids $\widehat{\mathrm{As}}$ and $x\mathbb{Q}[[x]]$. The proposition follows by taking invertible elements. □

**Proposition 7.** *The image of* $\exp^*$ *by* $\phi$ *is* $\exp x - 1$ *and the image of* $\log^*$ *is* $\log(1+x)$.

*Proof.* One has to prove that the coefficient of the linear tree with $n$ nodes in $\exp^*$ is $1/n!$ for all $n$. This can be done by recursion, using (11). □

This proposition explains what happens for linear vector fields. The linearity of a vector field $F$ on $\mathsf{A}^n$ implies that the pre-Lie algebra generated by $F$ inside $\mathsf{V}_n$ is associative, so that $\exp^*$ reduces to the standard exponential minus one.

## 4. Corollas and pointwise multiplication

There is another interesting kind of trees, opposite to linear trees. Let us call corollas the trees of depth no greater than two, where the depth is the maximum number of vertices in a chain of adjacent vertices starting from the root.

**Proposition 8.** *The subspace of* PreLie *spanned by labeled non-corollas is an ideal.*

*Proof.* Using the description above of the composition of PreLie, one shows that the depth of the composition of two labeled trees is greater or equal than the maximum of the depths of these labeled trees. Therefore, if one of the labeled trees has depth greater or equal to three, so does the composition. □

One can give a simple description of the quotient operad Mu. It has dimension $n$ in rank $n$ with basis given by the image of the labeled corollas with $n$ nodes. Let us call $\mu_i^n$ the image of the corolla with $n$ nodes and with root labeled by $i$ for $i = 1, \dots, n$.

Then $\mu_1^1$ is the unit of Mu and the composition is given by

$$\begin{cases} \mu_i^n \circ_i \mu_j^\ell = \mu_{i+j-1}^{n+\ell-1}, \\ \mu_i^n \circ_h \mu_j^\ell = 0 \text{ for } h \neq i \text{ and } \ell \geq 2. \end{cases} \tag{14}$$

Let $G_1$ be the group of formal power series of the form $1 + x\mathbb{Q}[[x]]$ for the pointwise multiplication product and $G_2$ be the multiplicative group $\mathbb{Q}^*$. There is an action of $G_2$ on $G_1$ by substitution: $\lambda \cdot f(x) = f(\lambda x)$. A group similar to the semi-direct product group $G_2 \ltimes G_1$ has been considered in [2, §2].

From the description of Mu above, one deduces that

**Proposition 9.** *The group* $\mathsf{G}_{\mathrm{Mu}}$ *is isomorphic to* $G_2 \ltimes G_1$.

*Proof.* The vector space $\mathrm{Mu}(n)_{\mathfrak{S}_n}$ is one-dimensional for all $n$, with basis given by the image of the corolla with $n$ nodes. Let us denote this basis element by $\nu_{n-1}$. Any element of $\mathsf{G}_{\mathrm{Mu}}$ can be uniquely written as the product $\lambda(\sum_{m\geq 0} f_m \nu_m)$ of $\lambda \in \mathbb{Q}^*$ and $f = \sum_{m \geq 0} f_m \nu_m$ with $f_0 = 1$. Let us compute the product of $\lambda f = \lambda(\sum_{m\geq 0} f_m \nu_m)$ and $\theta g = \theta(\sum_{n \geq 0} g_n \nu_n)$ with the conventions $f_0 = 1$ and $g_0 = 1$. One finds that

$$\lambda f \times \theta g = \sum_{m\geq 0}\sum_{n \geq 0} \lambda f_m \theta^m (\theta g_n) \nu_{n+m} = \lambda\theta \sum_{m\geq 0}\sum_{n\geq 0} \theta^m f_m g_n \nu_{n+m}. \tag{15}$$

One defines a map from $\mathsf{G}_{\mathrm{Mu}}$ to $G_2 \ltimes G_1$ by $\lambda(\sum_m f_m \nu_m) \mapsto (\lambda, f(x))$ with $f(x) = \sum_m f_m x^m$. The product in $G_2 \ltimes G_1$ is given by

$$(\lambda, f(x))(\theta, g(x)) = (\lambda\theta, f(\theta x) g(x)). \tag{16}$$

Hence the map is an isomorphism. □

Let us denote by $\psi$ the quotient map from PreLie to Mu.

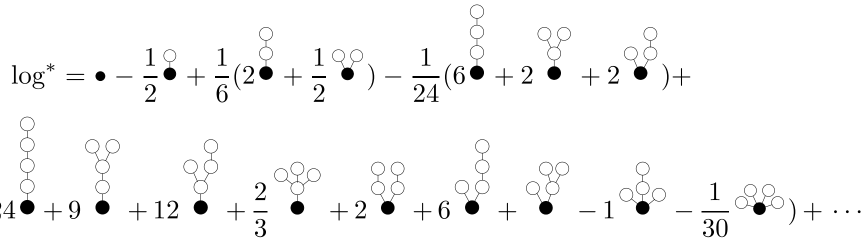
**Proposition 10.** *The image of* $\exp^*$ *by* $\psi$ *is* $(\exp x - 1)/x$ *and the image of* $\log^*$ *is* $x/(\exp x - 1)$.

*Proof.* One must prove that the coefficient of the corolla with $n$ nodes in $\exp^*$ is $1/n!$ for all $n$. The argument is a simple recursion using (11). $\square$

Therefore the coefficients of the corollas in $\log^*$ are related to the Bernoulli numbers, whose exponential generating function is precisely $x/(\exp x - 1)$.

## 5. Expansion

The coefficients of $\exp^*$, easily computed by Formula (11), are sometimes called the Connes-Moscovici coefficients [6]. There exists a direct procedure to compute the coefficient of any rooted tree in $\exp^*$, see [1, §2.2]. It may be interesting to find a way, other than inversion in the group $\mathsf{G}_{\mathrm{PreLie}}$, to compute the coefficients of $\log^*$. Here are the first terms of $\log^*$ in the rooted tree basis of $\widehat{\mathrm{PreLie}}$.

(17) 

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