Rogers dilogarithm and periodicity of cluster Y-systems

F. Chapoton

July 30, 2010

Plan

1. Some dilogarithm identities can be written in a special manner.
2. They are related to Y-systems and cluster algebras.
3. The cluster category lurks in the background.

Dilogarithms

First, the classical dilogarithm (Leonhard Euler)

$$\mathrm{Li}_2(x) = \sum_{n \geq 1} \frac{x^n}{n^2} = -\int_0^x \log(1-y)\frac{dy}{y}.$$

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Then, the Rogers dilogarithm (Leonard James Rogers)

$$\mathrm{L}(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x).$$

see reference book by Leonard Lewin.

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Famous **Spence-Abel identity**
with hidden cyclic symmetry of order 5.

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First two equations in a family indexed by $n \in \mathbb{N}$:

$$\sum_{f \in \mathbb{Y}_n} L(1/(1+f)) = n\,L(1),$$

where $\mathbb{Y}_n$ is a set of fractions in n variables.

$$\sum_{f \in \mathbb{Y}_n} \mathrm{L}(1/(1+f)) = n\,\mathrm{L}(1),$$

where

$$\mathbb{Y}_1 = \{x, 1/x\}$$

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Y-systems and cluster algebras

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In general, $\mathbb{Y}_n$ has $n + \binom{n+1}{2}$ elements.

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Since then, many interactions with other domains, including **Teichmüller theory**, Poisson geometry, representation theory, etc.

One important development is the **cluster category** due to Buan-Marsh-Reineke-Reiten-Todorov for the general acyclic case and to Caldero-Ch. in a special case,
with further developments by Claire Amiot.

Y-systems

Y-systems are sets of algebraic relations on variables $y_{i,n}$ that arose in the physics of conformal field theory (**Thermodynamic Bethe Ansatz**) in the works of Al. B. Zamolodchikov and many others.

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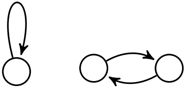
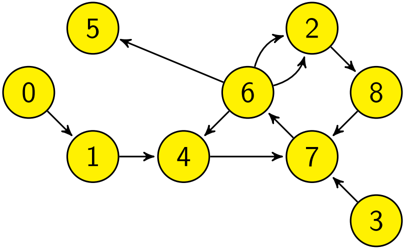
As shown by Fomin and Zelevinsky, Y-systems are closely related to cluster algebras.

Cluster algebras from quivers

Initial data for a cluster algebra: quiver $Q = (Q_0, Q_1)$

with no loops and no 2-cycles: 



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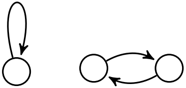
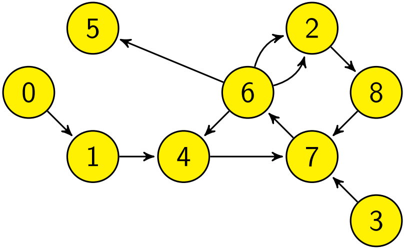
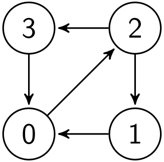
with no loops and no 2-cycles: 



One considers all quivers obtained from the initial one by **mutation of quivers**.

Mutation of quivers

One can mutate a quiver Q at any vertex $i \in Q_0$.



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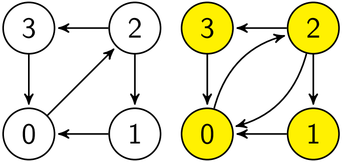

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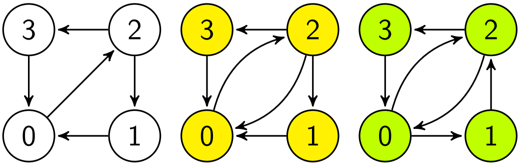

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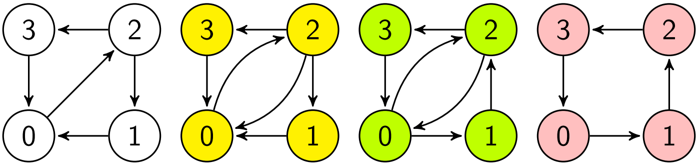

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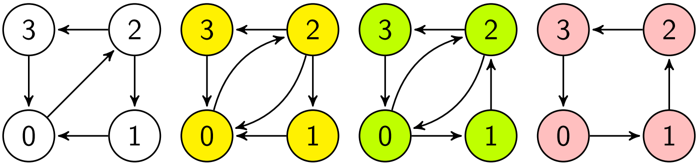
This defines a new quiver $\mu_i(Q)$ and $\mu_i(\mu_i(Q)) = Q$.



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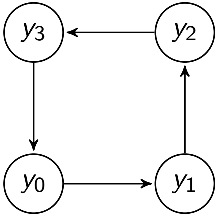
A Y-**seed** is a pair consisting of a quiver with a fraction (in the variables y_i) attached to every vertex.



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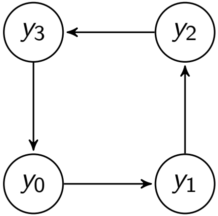

The initial Y-seed

Then consider all the seeds obtained from the initial one by **mutation of** Y**-seeds**.

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$$y'_j = \begin{cases} 1/y_i & \text{if} \quad j = i, \\ y_j \prod_{i \to j}(1 + y_i) / \prod_{j \to i}(1 + 1/y_i) & \text{if} \quad j \neq i. \end{cases}$$

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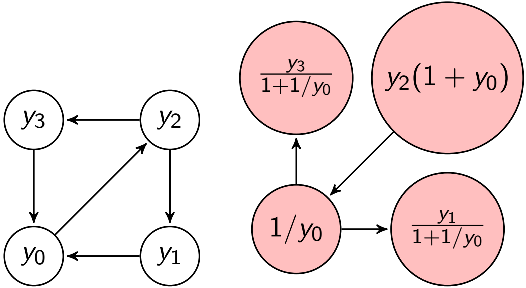

Q and $\mu_0(Q)$

Finite Y-systems

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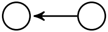
For example, for the quiver  of type $\mathbb{A}_2$, with initial seed (y_1, y_2) one gets the following set of Y-variables:

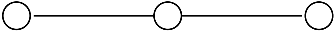
$$y_1, y_2, y_1(1 + y_2), (1 + y_1 + y_1 y_2)/y_2, \frac{y_1 + 1}{y_1 y_2}$$

and their inverses

$$1/y_1, 1/y_2, \frac{1}{y_1(1 + y_2)}, \frac{y_2}{1 + y_1 + y_1 y_2}, \frac{y_1 y_2}{y_1 + 1}.$$

Precise setting for dilogarithm identities

Fix a Dynkin diagram of type $\mathbb{A}, \mathbb{D}, \mathbb{E}$. For example $\mathbb{A}_3$.



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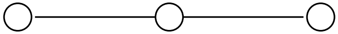

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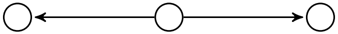
$Q =$ 

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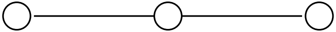

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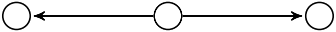
$Q =$ 

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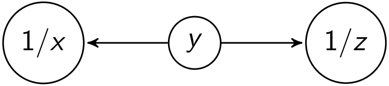

(variables y_i for sources, inverses $1/y_i$ for sinks)

Restricted Y-system

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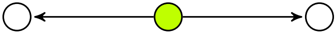
$Q =$ 

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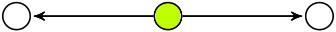
$Q =$ 

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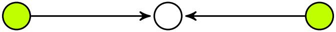
$\mu_+(Q) =$ 

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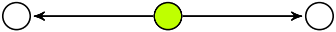

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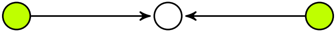

Let μ_- be the composition of all mutations at the sources of $\mu_+(Q)$. One has $\mu_-(\mu_+(Q)) = Q$.

Restricted Y-system

Theorem (Fomin & Zelevinsky)

The composition $\mu_- \circ \mu_+$ acting on Y-seeds is periodic. Let $\mathbb{Y}$ be the subset of the Y-variables that correspond to sources in the Y-seeds obtained by iterating μ_+ and μ_-. The set $\mathbb{Y}$ is made of Laurent polynomials in the variables y_i. The non-trivial denominators are in natural bijection with positive roots for the Dynkin diagram.

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For example, one gets in type $\mathbb{A}_3$

$$\color{red}{x, y, z}, \frac{x+z+xz+1}{y}, \frac{x+2y+z+y^2+xy+xz+yz+1}{xyz}$$

$$\frac{x+y+z+xz+1}{xy}, \frac{x+y+z+xz+1}{yz}, \frac{y+1}{x}, \frac{y+1}{z}.$$

Identity associated with a Dynkin diagram

Theorem (Ch.)

Let X_n be a simply-laced Dynkin diagram of rank n. Let $\mathbb{Y}(X_n)$ be the subset of the Y-system described before. Then

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Type $\mathbb{D}_n$ gives another family of identities. Three other identities for $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$.

About the proof

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To compute the value of the sum, one takes the limit where all variables go to 0. Among elements of $\mathbb{Y}$, exactly n have limit 0 (hence add L(1) to the sum) and the others have limit $+\infty$ (hence do not contribute to the sum). For example in type $\mathbb{A}_3$:

$$x, y, z, \frac{x+z+xz+1}{y}, \frac{x+2y+z+y^2+xy+xz+yz+1}{xyz}$$
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Dilogarithm identities associated to pairs of Dynkin diagrams, that generalize the identities presented here, have been proved by Nakanishi, using the periodicity theorem due to Keller.

More general identities have been obtained in recent works by Inoue, Iyama, Keller, Kuniba and Nakanishi.

They use similar methods, namely the relation to cluster algebras and cluster categories.

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Let Q be a quiver with no cycles.

A **representation** of Q is given by

vector spaces M_i corresponding to vertices $i \in Q_0$ of Q,

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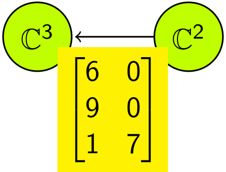
For example, a representation of the quiver of type $\mathbb{A}_2$ is



The **dimension vector** of M is $\dim M = (\dim M_i)_{i \in Q_0}$.

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Very similar to usual Grassmannian. Can be defined as a subvariety of a product of classical Grassmannians.

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Introduce now the following polynomials in variables $(x_i)_i$:

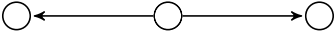
$$F_M = \sum_{e \leq m} \chi(\mathrm{Gr}_e(M)) \prod_i {x_i}^{e_i},$$

where χ is the Euler characteristic.

Call them the F-**polynomials**.

Example of F-polynomial

Consider for example the quiver

$Q =$ 

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Therefore the F-polynomial for this representation is

$$1 + x_1 + x_3 + x_1 x_3 + x_1 x_2 x_3.$$

Fact

The polynomials F_M are sufficient to recover Y-variables.

In fact, the Y-variables are products and quotients of F-polynomials and monomials (There are explicit formulas...).

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One can also use the same F-polynomials to compute **cluster variables** (not defined in this talk, but even more important).

Cluster category

Cluster variables and Y-variables are closely related to F-polynomials, which are defined using the category mod Q of representations of the quiver Q.

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The cluster category $\mathcal{C}_Q$ is defined as a kind of quotient of the derived category $D \operatorname{mod} Q$ of the category $\operatorname{mod} Q$ in which the shift functor [1] becomes isomorphic with the Auslander-Reiten functor τ. It remains a triangulated category.

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This definition works for acyclic quivers. More and more general versions are available, for quivers with potentials, in works of Bernhard Keller, Yann Palu, Claire Amiot, Pierre-Guy Plamondon and others.

In the category $D \bmod Q$, the Auslander-Reiten functor satisfies

$$\tau^h = [-2], \tag{1}$$

where h is the Coxeter number.
Therefore in the cluster category $\mathcal{C}_Q$, where $\tau = [1]$, one has

$$[h+2] = \mathrm{Id}\,. \tag{2}$$

This means that the shift functor is periodic of period $h+2$.

For example, this explains the period 5 for the quiver $\mathbb{A}_2$, for which $h=3$.

Clusters and geometry of surfaces

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Final remarks

Clusters and geometry of surfaces

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Quantum clusters

There is an obvious way to define quantum F-polynomials: replace the Euler characteristic by the number of points over finite field $\mathbb{F}_q$. They can probably be used to define quantum Y-systems.

Question: can one use these in quantum dilogarithm identities?

Post-scriptum

Advertisment

(Nothing to do with the talk)

arXiv:0909.1694

a note on q-Zeta operators and Bernoulli-Carlitz fractions