Introduction to operads

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Abstract

This paper is a written version of a lecture given at the conference "Hopf algebras, renormalization and the Riemann-Hilbert problem" at Luminy. It contains a short introduction to operads.

Brief historical introduction

The notion of operad has its roots in the realm of algebraic topology. It can be tracked to the work of Stasheff and Boardman-Vogt in the 60's and has been formally introduced in 1972 by May. The original motivation was the study of loop spaces and iterated loop spaces.

Operads have gained a new interest in the 90's with the use of several new kinds of algebras in mathematical physics. For example, the moduli spaces of Riemann surfaces, which appear more or less explicitly in conformal field theory, form an operad of spaces.

There exist several good introductory articles on operads : see [6] for an historical account, [3, 4] for survey articles and [1, 2] for more technical material.

1 Operads : what for ?

In a certain sense, operads arise naturally as generalizations of associative algebras. One can make the following comparison.

The notion of associative algebra arise from the study of properties of the composition of endomorphisms of a vector space. Given several endomorphisms, it is useful to study the algebra they generate. This leads directly to representation theory, which is the study of maps from an abstract associative algebra A to $\mathrm{end}(V)$ for some vector space V.

Similarly, the notion of operad appears when one considers the composition of multi-linear maps from a vector space to itself. Given a set of multi-linear maps, one can study disjointly the operad they generate and the ways this operad can be realized by multi-linear maps.

The analog of a representation of an associative algebra A is named an algebra over an operad $\mathcal{P}$. For each vector space, one has an operad $\mathcal{E}\mathrm{nd}(V)$ made of multi-linear maps. A structure of algebra over $\mathcal{P}$ on the vector space V is nothing but a map of operads from $\mathcal{P}$ to $\mathcal{E}\mathrm{nd}(V)$.

2 Definition and examples

For concreteness, one starts with the category of vector spaces. The set $\{1, 2, \dots\}$ is denoted by $\mathbb{N}$.

An operad $\mathcal{P}$ is the data of

- a collection of vector spaces $\mathcal{P}(n)$ for $n \in \mathbb{N}$,
- actions of the symmetric groups $\mathfrak{S}_n$ on $\mathcal{P}(n)$,
- a distinguished element I of $\mathcal{P}(1)$ (called unit),
- composition maps $\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1)$ for $m, n \in \mathbb{N}$ and $i = 1, \dots, m$.

These are subject to the following relationships (let x, y, z be in $\mathcal{P}(m), \mathcal{P}(n), \mathcal{P}(p)$ respectively) :

- Equivariance : compatibility of the symmetric groups actions with the compositions,
- Associativity I (Disjoint composition): $(x \circ_i y) \circ_j z = (x \circ_j z) \circ_i y$ for all x, y, z and $i \neq j$ in $\{1, \dots, m\}$,
- Associativity II (Nested composition): $(x \circ_i y) \circ_j z = x \circ_i (y \circ_j z)$ for all x, y, z, i in $\{1, \dots, m\}$ and j in $\{1, \dots, n\}$,
- Unit : $I \circ_1 x = x = x \circ_i I$ for all m, $i = 1, \dots, m$.

Example 1 : As explained before, the main example is the operad $\mathcal{E}\mathrm{nd}(V)$ of multi-linear maps of a vector space. The vector space $\mathcal{E}\mathrm{nd}(V)(n)$ is defined by $\hom(V^{\otimes n}, V)$. The action of the symmetric groups $\mathfrak{S}_n$ is by permutations of tensors on the source $V^{\otimes n}$ of the maps. The composition $\circ_i$ is the usual composition of multi-linear maps :

$$(f \circ_i g)(x_1, \dots, x_{m+n-1}) = f(x_1, \dots, x_{i-1}, g(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{n+m-1}).$$

The unit is the identity map I_V.

Example 2 : One takes for $\mathrm{Com}(n)$ the trivial representation of $\mathfrak{S}_n$ with basis e_n. The composition is given by $e_m \circ_i e_n = e_{n+m-1}$ and the unit is e_1. The algebras over this operad are exactly the commutative associative algebras.

Example 3 : One takes $\mathrm{As}(n)$ to be the regular representation of $\mathfrak{S}_n$ with a basis indexed by permutations, seen as words in the letters $\{1, \dots, n\}$, and the action given by changing the names of the letters. The composition $\tau \circ_i \sigma$ of a permutation $\sigma \in \mathfrak{S}_n$ inside the letter i of a permutation $\tau \in \mathfrak{S}_m$ is given by insertion and renumbering. For example, one has

$$(1342) \circ_2 (213) \rightsquigarrow (134(213)) \rightsquigarrow (156324), \tag{1}$$

where the second operation consists in adding $n-1$ to the numbers in τ which are strictly greater than i and adding $i-1$ to the numbers in σ. The unit is the identity $1 \in \mathfrak{S}_1$. The algebras over the operad As are exactly the associative algebras.

Remarks :

If one forgets the symmetric groups actions in the axioms of an operad, one gets the notion of non-symmetric operad, which is the one Stasheff used to study homotopy theory of based loop spaces.

There is an alternative definition which uses multiple composition maps and is equivalent to the definition above in presence of a unit, see [5] for details.

3 Free algebras

Given any operad $\mathcal{P}$, one can define the free $\mathcal{P}$-algebra over a vector space V. The underlying vector space is

$$\mathsf{F}_{\mathcal{P}}(V) = \bigoplus_{n \in \mathbb{N}} \left(V^{\otimes n} \otimes \mathcal{P}(n)\right)_{\mathfrak{S}_n},$$

where $()_{\mathfrak{S}_n}$ means taking the coinvariants of the action of $\mathfrak{S}_n$. To define the structure of a $\mathcal{P}$-algebra on $\mathsf{F}_{\mathcal{P}}(V)$, one needs a map from $\mathcal{P}$ to $\mathcal{E}\mathrm{nd}(\mathsf{F}_{\mathcal{P}}(V))$. This is defined using the composition maps of the operad $\mathcal{P}$.

One can recover the operad from the free algebras. More precisely, one can identify $\mathcal{P}(n)$ as the subspace of $\mathsf{F}_{\mathcal{P}}(\{x_1, \dots, x_n\})$ which is of degree $(1, \dots, 1)$ with respect to $(x_1, \dots, x_n)$. Then the substitution of tensors gives composition maps.

Example : The free associative algebra (without unit) on a vector space V is given by $\mathsf{T}(V) = \oplus_{n \geq 1} V^{\otimes n}$. The subspace of degree $(1 \dots, 1)$ of $\mathsf{T}(\{x_1, \dots, x_n\})$ is the regular representation of $\mathfrak{S}_n$ as described in the example 2 above. The previous description of the operad As can be derived from the substitution of tensors.

4 Generators and relations

The following definitions are natural analogues of the case of associative algebras.

A *morphism* of operads f from $\mathcal{P}$ to $\mathcal{Q}$ is a collection of equivariant maps f_n from $\mathcal{P}(n)$ to $\mathcal{Q}(n)$ which are assumed to be compatible with the compositions and which maps unit to unit.

An *ideal* of an operad $\mathcal{P}$ is a collection $\mathcal{I}$ of $\mathfrak{S}_n$-submodules $\mathcal{I}(n) \subset \mathcal{P}(n)$ such that the composition of an element of $\mathcal{I}$ inside one of $\mathcal{P}$ is in $\mathcal{I}$, as well as the composition of an element of $\mathcal{P}$ inside one of $\mathcal{I}$.

If one is given a collection of vector spaces $\mathcal{E}(n)$ for $n \in \mathbb{N}$, one can define the free operad on $\mathcal{E}$, denoted $F(\mathcal{E})$. This construction uses trees in an essential way. More precisely, $F(\mathcal{E})(n)$ is generated by (rooted) trees with elements of $\mathcal{E}$ at each vertex and the composition is the grafting of trees.

One can now introduce the notion of presentation of an operad by generators and relations, in analogy with presentations of associative algebras.

Let $\mathcal{E}$ be a collection of vector spaces $\mathcal{E}(n)$ for $n \in \mathbb{N}$ and $\mathcal{R}$ be a sub-collection $\mathcal{R}(n) \subset F(\mathcal{E})(n)$. One defines the operad generated by $\mathcal{E}$ and the relations $\mathcal{R}$ to be the quotient of $F(\mathcal{E})$ by the ideal generated by $\mathcal{R}$.

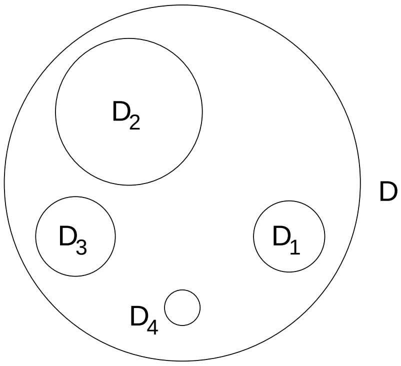

Figure 1: An element of the little discs operad

In general, the operad describing a type of algebra has a generator for each product of this type of algebra and a relation for each axiom satisfied by these products.

EXAMPLE 1 : The operad As describing associative algebras is generated by the regular representation As(2) of $\mathfrak{S}_2$ (corresponding to the associative product) and the relation of associativity $12 \circ_1 12 = 12 \circ_2 12$ in $F(\mathrm{As}(2))(3)$.

EXAMPLE 2 : The operad Com describing commutative associative algebras is generated by the trivial representation of $\mathfrak{S}_2$ (corresponding to the commutative product) and the relation of associativity in $F(\mathrm{Com}(2))(3)$.

EXAMPLE 3 : The operad Lie describing Lie algebras is generated by the sign representation of $\mathfrak{S}_2$ (antisymmetric product) and the Jacobi relation in $F(\mathrm{sgn}_2)(3)$.

There seems not to exist a simple combinatorial description of the Lie operad similar to the one for As. In fact, one of the known descriptions is by the mean of configuration spaces.

5 Operads of spaces

One can define the notion of operad in many others categories, such as Sets, $\mathsf{TopologicalSpaces}$, $\mathsf{Manifolds}$, $\mathsf{Complexes}$, $\mathsf{HodgeStructures}$ and so on.

The little disc operad $\mathcal{D}$ is a simple example of an operad of topological spaces.

The space $\mathcal{D}(n)$ is the set of embeddings of n disjoint labelled discs inside the disc D of radius one and center zero in $\mathbb{R}^2$. It can be seen as a sub-variety of the product $\left(\mathsf{D} \times \mathbb{R}_+^*\right)^n$ by using as coordinates the centers and radiuses of the n discs.

The composition maps are given by scaling, translation and insertion. The composition $\mathsf{D} \circ_i \mathsf{D}'$ is described as follow. The disc D' is scaled by the factor r_i (radius of the i^{th} disc of D) and substituted for the i^{th} disc of D, which is erased.

A more sophisticated example is the operad formed by the moduli spaces of stable curves (i.e. stable Riemann surfaces). If one restricts to genus zero curves, this space parameterize certain trees of $\mathbb{C}P^1$ with $n+1$ marked points indexed by $\{0, \ldots, n\}$. The composition is given by gluing, which means identifying two points.

Given an operad of topological spaces, one can apply an homology functor to get an algebraic operad in the category of graded vector spaces.

This is a useful way to build operads. For example, the homology of the little discs operad $\mathcal{D}$ is the so-called Gerstenhaber operad G, which describes some kind of twisted graded Poisson algebras. In particular, this operad contains the operad Lie.

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