Rooted trees, pre-Lie algebras and vector fields

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Abstract

This paper is a written version of a lecture given at the conference "Hopf algebras, renormalization and the Riemann-Hilbert problem" at Luminy. It deals with the relationship between pre-Lie algebras, rooted trees and vector fields.

1 Pre-Lie algebras

Pre-Lie algebras have been known since the 60's, sometimes under the following names : left-symmetric algebras, right-symmetric algebras, Vinberg algebras or quasi-associative algebras. They have appeared in the study of affine structures on manifolds [5] and in deformation theory [4].

A Pre-Lie algebra is a vector space V endowed with a bilinear multiplication $\curvearrowleft : V \otimes V \to V$ such that

$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y) \tag{1}$$

for all $x, y, z \in V$.

Remarks :

One can see (1) as a condition of invariance under the swap of y and z.

In the case the product is associative, both sides of (1) are zero, so an associative algebra is also a pre-Lie algebra.

One can define an antisymmetric product on a pre-Lie algebra by the commutator $[x, y] := x \curvearrowleft y - y \curvearrowleft x$. It follows from (1) that this bracket satisfies the Jacobi identity. This Lie algebra is denoted by V_{Lie}.

Furthermore, one can then use (1) to show that the map $\curvearrowleft : V \otimes V_{\mathrm{Lie}} \to V$ defines a right action of V_{Lie} on V.

Formula (1) also implies that if a pre-Lie product is commutative, then it is in fact associative.

2 Combinatorial examples

Given objects which can be nested one inside another in a finite number of different ways, one can define a pre-Lie algebra. For example, Feynman diagrams can under certain conditions be inserted at vertices or edges of other Feynman diagrams. This is how pre-Lie algebras appear in renormalization [2, 3]. The way this construction works is made clear with the following examples.

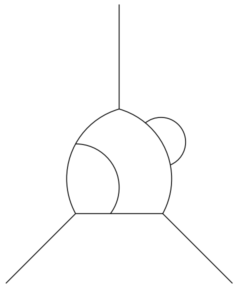

Figure 1: An example of trivalent graph.

Example 1 : Trivalent graphs.

Let $\mathcal{T}$ be the set of pairs (G, v_G) where G is a trivalent graph with vertex set $V(G)$ and $v_G \in V(G)$ a distinguished vertex such that $G \setminus v_G$ is connected. One can picture such a graph as a sort of Feynman diagram with three legs, see figure 2.

Let T be the vector space generated by $\mathcal{T}$. One defines a pre-Lie product on T by

$$G \curvearrowleft H = \sum_{v \in V(G) \setminus v_G} G \curvearrowleft^v H, \tag{2}$$

where $G \curvearrowleft^v H$ is the sum of the six graphs obtained by gluing the three legs of H to the three edges of the (removed) vertex v of G in all possible ways.

Then one shows easily that

$$(G \curvearrowleft H) \curvearrowleft K - G \curvearrowleft (H \curvearrowleft K) \tag{3}$$

is given by the same kind of insertion of H and K in two different vertices of G. This is symmetric in H and K, so we get a pre-Lie algebra.

Example 2 : Words in two letters.

Let us consider the set of finite words on the alphabet $\{A, B\}$ beginning with A and ending with B, such as AAB or $ABABBAB$. If w is such a word, let us denote by $w[i]$ the i^{th} letter of w and by $|w|$ the length of w.

Let W be the vector space generated by this set of words. One define a pre-Lie product on W by the formula

$$w_1 \circ w_2 = \sum_{i=0}^{|w_1|} \varepsilon(i)\, w_1 \curvearrowleft^i w_2, \tag{4}$$

where $w_1 \curvearrowleft^i w_2$ is the insertion of w_2 between $w_1[i]$ and $w_1[i+1]$ and $\varepsilon(i)$ is defined by

$$\begin{cases} -1 \text{ if } w_1[i] = A \text{ and } w_1[i+1] = B \\ +1 \text{ if } w_1[i] = B \text{ or } \emptyset \text{ and } w_1[i+1] = A \text{ or } \emptyset \\ 0 \text{ else.} \end{cases} \tag{5}$$

For example, $ABAB \curvearrowleft AB = 3ABABAB - AABBAB - ABAABB$.

Taking care of signs, one can see that this is a pre-Lie product. It appears in the study of multiple zeta values [6].

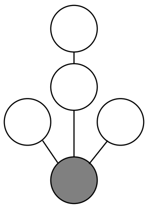

Figure 2: An example of rooted tree.

Example 3 : Labeled rooted trees

A rooted tree is a pair (T, v) where T is a finite connected and simply connected graph and v is a distinguished vertex of T called the root, see figure 2.

Let S be a fixed finite set. A labeling of a tree T by S is a map from S to the vertex set $V(T)$ of T. Let $\mathrm{RT}(S)$ be the vector space generated by the set of S-labeled rooted trees. One defines a pre-Lie product on $\mathrm{RT}(S)$ by

$$T_1 \curvearrowleft T_2 = \sum_{v \in V(T_1)} T_1 \curvearrowleft^v T_2, \tag{6}$$

where $T_1 \curvearrowleft^v T_2$ is the rooted tree defined by adding to the disjoint union of T_1 and T_2 an edge joining the root of T_2 to the vertex v of T_1, and keeping the root of T_1 as root vertex.

Here again the expression $(T_1 \curvearrowleft T_2) \curvearrowleft T_3 - T_1 \curvearrowleft (T_2 \curvearrowleft T_3)$ can be described as grafting both T_2 and T_3 to vertices of T_1. This is symmetric in T_2, T_3, so the pre-Lie axiom is checked.

Example 4 : Operads

Let $\mathcal{P}$ be an operad of vector spaces. There's a pre-Lie product on the graded vector space $\mathsf{P} = \oplus_{n \in \mathbb{N}} \mathcal{P}(n)$ defined for $x \in \mathcal{P}(m)$ and $y \in \mathcal{P}(n)$ by

$$x \curvearrowleft y = \sum_{i=1}^{m} x \circ_i y, \tag{7}$$

where $\circ_i$ are the composition maps of the operad $\mathcal{P}$.

The axioms of Associativity I and II of an operad ensure that this is a pre-Lie product.

3 The underlying operad

In fact, the third example is very special : it is free as a pre-Lie algebra. The operad PreLie is indeed described in terms of rooted trees [1].

Let $\mathrm{PreLie}(n)$ be the vector space generated by rooted trees T endowed with a bijection $\{1, \dots, n\} \simeq V(T)$. The edges are oriented toward the root.

The composition $T_1 \circ_i T_2$ of $T_2 \in \mathrm{PreLie}(n)$ in the vertex i of $T_1 \in \mathrm{PreLie}(m)$ is given by the following description.

Consider the tree T_1 with its vertex i drawn as a big circle. Put inside this circle the tree T_2. Let $In(i)$ be the incoming edges of the vertex i. Given a map f from $In(i)$ to $V(T_2)$, one can define a new tree $T_1 \circ_i^f T_2$ by erasing the big circle, grafting the formerly incoming edges to vertices of T_2 and the formerly outgoing edge to the root of T_2. Then

$$T_1 \circ_i T_2 = \sum_{f : In(i) \to V(T_2)} T_1 \circ_i^f T_2. \tag{8}$$

From this description of the pre-Lie operad, one can recover the description of the free pre-Lie algebras on a set S given in the example 3.

4 Affine structures, vector fields

Pre-Lie algebras have a completely different geometrical origin. Given a manifold with an affine structure (atlas with affine change-of-coordinates map), the Lie algebra of vector fields can be seen to come from a pre-Lie algebra structure. For example, this works for open sets in affine spaces or torus.

In the case of polynomial vector fields on $\mathbb{R}$, the product is given by

$$P(x)\frac{d}{dx} \curvearrowleft Q(x)\frac{d}{dx} = Q\frac{dP}{dx}\frac{d}{dx}. \tag{9}$$

This is not much more complicated for $\mathbb{R}^n$:

$$\sum_i P_i \partial_i \curvearrowleft \sum_j Q_j \partial_j = \sum_{i,j} Q_j \left(\partial_j P_i\right) \partial_i. \tag{10}$$

The pre-Lie product is useful to solve the flow equations of vector fields. Consider the flow equation for a function $X : \mathbb{R} \to \mathbb{R}^n$:

$$\frac{dX}{dt} = F(X(t)) \text{ and } X(0) = X_0. \tag{11}$$

Then the Taylor expansion of $X(t)$ is given by the formula

$$X(t) = X_0 + \sum_{n \geq 1} F^{\curvearrowleft n}(X_0)\frac{t^n}{n!}, \tag{12}$$

where $F^{\curvearrowleft n}$ is the n^{th} right iterate of F for the pre-Lie product on vector fields.

The formal power series

$$F \mapsto \sum_{n \geq 1} \frac{F^{\curvearrowleft n}}{n!} \tag{13}$$

deserves to be called the "pre-Lie exponential". It becomes the function $x \mapsto \exp(x) - 1$ if the pre-Lie product is in fact associative.

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