The surjectivity of the combinatorial Laplacian on infinite graphs

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A simplicial graph is a pair $G = (V, \sim)$, where

- V is a nonempty set whose elements are the vertices of the graph ;
- $\sim$ is a relation on V such that
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• A graph is called locally finite if each of its vertices has only finitely many neighbor vertices.

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$$\forall f \in \mathbb{R}^V, \forall v \in V, \quad \Delta_G(f)(v) = f(v) - \frac{1}{\deg(v)} \sum_{v \sim w} f(w) = \frac{1}{\deg(v)} \sum_{v \sim w} \left(f(v) - f(w) \right).$$

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Two vertices v and w of the graph $G = (V, \sim)$ are in the same connected component if there exists a finite sequence of vertices $v = u_0, u_1, \ldots, u_n = w$ with $u_k \sim u_{k+1}$ for $0 \leq k \leq n-1$.

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If $G_i = (V_i, \sim)$ are the connected components of the graph G, one has

$$\mathbb{R}^V = \prod_i \mathbb{R}^{V_i} \quad \text{and} \quad \Delta_G = \prod_i \Delta_{G_i}.$$

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The graph G is called connected if it has only one connected component.

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Indeed, for all $f, g \in \mathbb{R}^V$, we have

$$\begin{aligned}\langle f, \Delta_G(g) \rangle &= \sum_{v \sim w} f(v)\left(g(v) - g(w)\right) \\ &= \frac{1}{2}\left(\sum_{v \sim w} f(v)\left(g(v) - g(w)\right) + \sum_{v \sim w} f(w)\left(g(w) - g(v)\right)\right),\end{aligned}$$

and hence

$$\langle f, \Delta_G(g) \rangle = \frac{1}{2} \sum_{v \sim w} (f(v) - f(w))(g(v) - g(w)) = \langle \Delta_G(f), g \rangle.$$

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In particular, we have

$$\forall f \in \mathbb{R}^V, \quad \langle f, \Delta_G(f) \rangle = \frac{1}{2} \sum_{v \sim w} (f(v) - f(w))^2.$$

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It follows that if G is finite and connected, then the kernel of Δ_G is reduced to the constant functions (this may also be established by means of the maximum principle) and its image is

$$\operatorname{Im} \Delta_G = (\operatorname{Ker} \Delta_G)^{\perp} = \{f \in \mathbb{R}^V : \sum_{v \in V} \deg(v) f(v) = 0\}.$$

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We have

$$\Delta_G(f)(n) = \begin{cases} f(0) - f(1) & \text{if } n = 0, \\ f(n) - \frac{f(n-1)+f(n+1)}{2} & \text{if } n \geq 1. \end{cases}$$

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The kernel of Δ_G is reduced to the constant functions and Δ_G is surjective.

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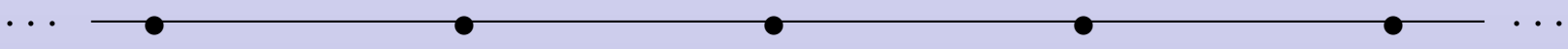

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The kernel of Δ_G is 2-dimensional (arithmetic sequences $n \mapsto C_1 + C_2 n$) and Δ_G is surjective.

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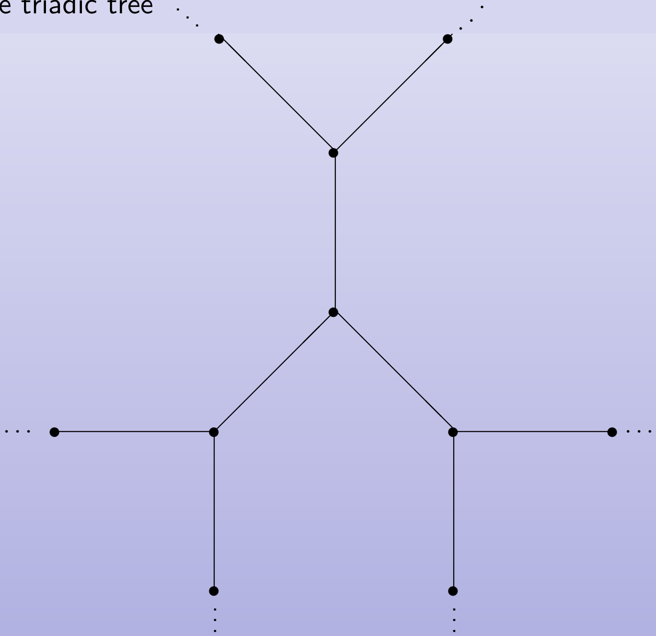

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With each end $\omega \in \Omega$, one can associate a harmonic function h_ω defined in the following way. We first put $h_\omega(v_0) = 1$. Then, starting from v_0, the value taken by h_ω is multiplied by 2 when we go to a neighbor vertex in the direction of the end, and divided by 2 when we go to a neighbor vertex in a direction which is not that of the end.

Example: The triadic tree (3)

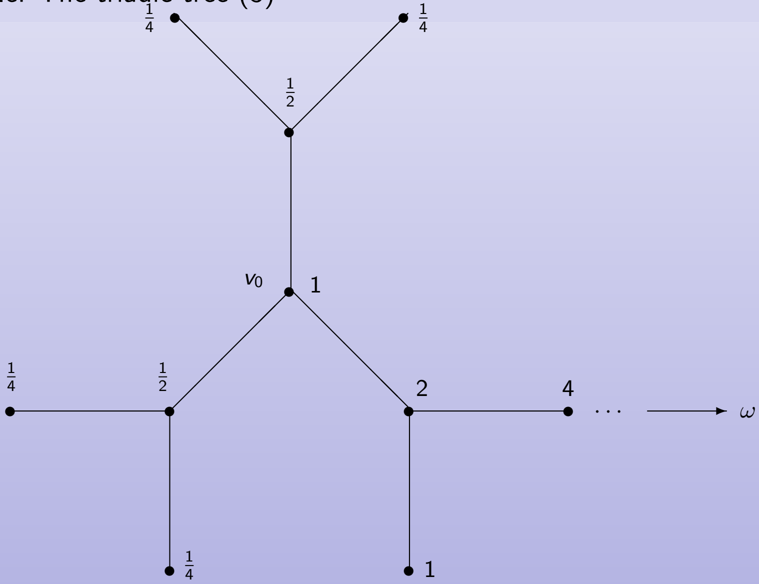

Figure 5: Harmonic function associated with an end

Example : The triadic tree (4)

Figure 6: $h_\omega(v) = 2^{b-a}$

We have

$$\forall \omega, \omega' \in \Omega, \quad \lim_{v \to \omega'} h_\omega(v) = \begin{cases} +\infty & \text{if } \omega' = \omega, \\ 0 & \text{if } \omega' \neq \omega. \end{cases}$$

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Thus, the kernel of Δ_G has uncountable dimension.
Here again, one easily checks that Δ_G is surjective.

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There exist infinite 3-regular graphs where all harmonic functions are constant (cf. [Tro-1998]) as the following one :

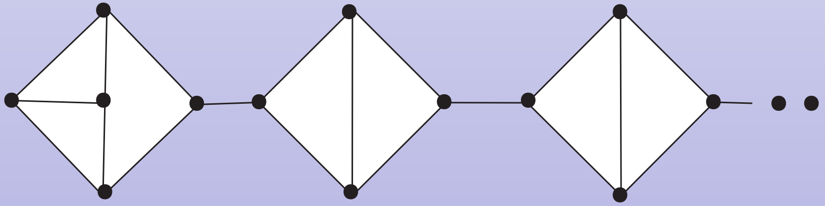

Figure 7: A 3-regular graph

This is joint work with Tullio Ceccherini-Silberstein (Rome) and Józef Dodziuk (NYC).

Theorem (CCD-2011)

Let $G = (V, \sim)$ be a connected, infinite, locally finite, simplicial graph. Then the combinatorial Laplacian $\Delta_G \colon \mathbb{R}^V \to \mathbb{R}^V$ is surjective.

Surjectivity of the Laplacian

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Let $G = (V, \sim)$ be a connected, infinite, locally finite, simplicial graph. Then the combinatorial Laplacian $\Delta_G \colon \mathbb{R}^V \to \mathbb{R}^V$ is surjective.

Plan of the proof:
We equip $\mathbb{R}^V$ with its prodiscrete topology and we establish the following points:

- $\operatorname{Im} \Delta_G$ is dense in $\mathbb{R}^V$,
- $\operatorname{Im} \Delta_G$ is closed in $\mathbb{R}^V$,

which imply $\operatorname{Im} \Delta_G = \mathbb{R}^V$.

The prodiscrete topology on $\mathbb{R}^V$

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The prodiscrete topology on $\mathbb{R}^V$ is metrizable. Indeed, let us fix a vertex v_0 and denote by B_n the set of vertices at distance $\leq n$ from v_0 (i.e., that can be joined to v_0 by a chain of neighbor vertices of length $\leq n$). We have $V = \bigcup_{n \in \mathbb{N}} B_n$ and $B_n \subset B_{n+1}$ for all n. Then

$$\forall f, g \in \mathbb{R}^V, \quad d(f,g) = \sum_{n \in \mathbb{N}} \frac{\alpha_n}{2^{n+1}},$$

where $\alpha_n = 0$ if f and g coincide on B_n and $\alpha_n = 1$ otherwise, is a metric on $\mathbb{R}^V$ compatible with the prodiscrete topology.

Proof of the density of the image

Let $F_n \subset \mathbb{R}^V$ denote the vector subspace consisting of all functions $f \in \mathbb{R}^V$ whose support is contained in B_n.
Consider the linear map $\delta_n \colon F_n \to F_n$ defined by

$$\forall f \in F_n, \forall v \in V, \quad \delta_n(f)(v) = \begin{cases} \Delta_G(f)(v) & \text{if } v \in B_n \\ 0 & \text{otherwise.} \end{cases}$$

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The maximum principle implies that δ_n is injective. Indeed, if $f \in \operatorname{Ker} \delta_n$, we have

$$\forall v \in B_n, \quad |f(v)| = \left| \frac{1}{\deg(v)} \sum_{v \sim w} f(w) \right| \leq \frac{1}{\deg(v)} \sum_{v \sim w} |f(w)|.$$

Consequently, if $v \in B_n$ satisfies $|f(v)| = M = \max |f|$, then $|f(w)| = M$ for all $w \in V$ such that $v \sim w$. It follows that $|f|$ is constant on B_{n+1}.
As V is infinite, we can find a vertex which is in B_{n+1} but not in B_n. This shows that f is identically 0.

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As F_n is finite-dimensional, the injectivity of δ_n implies its surjectivity. Therefore, for all $g \in \mathbb{R}^V$ and $n \in \mathbb{N}$, we can find $f \in F_n$ such that $\Delta_G(f)$ coincide with g on B_n. This shows that $\operatorname{Im} \Delta_G$ is dense in $\mathbb{R}^V$.

Projective sequences of sets

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This projective limit may be empty even if all the sets X_n are nonempty.

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$$X_0 \xleftarrow{u_0} X_1 \xleftarrow{u_1} X_2 \xleftarrow{u_2} X_3 \xleftarrow{u_3} \dots$$

The projective limit of the projective sequence (X_n, u_n) is the set $\varprojlim(X_n, u_n)$ consisting of all sequences $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in X_n$ and $x_n = u_n(x_{n+1})$ for all $n \in \mathbb{N}$.
This projective limit may be empty even if all the sets X_n are nonempty.
However, if all the maps u_n are surjective and $X_0 \neq \varnothing$, then we clearly have $\varprojlim(X_n, u_n) \neq \varnothing$ (take consecutive preimages of an element $x_0 \in X_0$).

A Mittag-Leffler-type lemma

Lemma 1

Let $(X_n, u_n)_{n \in \mathbb{N}}$ *be a projective sequence of sets and let* K *be a field. Suppose that each* X_n *is a nonempty finite-dimensional affine space over* K *and that all the maps* $u_n \colon X_{n+1} \to X_n$ *are affine. Then one has* $\varprojlim (X_n, u_n) \neq \varnothing$.

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Proof.

For $n \leq m$, define $\gamma_{nm} \colon X_m \to X_n$ by

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It follows that $\varnothing \neq \varprojlim(X'_n, u'_n) = \varprojlim(X_n, u_n)$. □

A Mittag-Leffler-type lemma (2)

Remark

The preceding lemma becomes false if the hypothesis saying that the affine spaces X_n are finite-dimensional is removed. Indeed, consider for example the Hilbert space

$$\ell^2(\mathbb{N}) = \{(a_i)_{i \in \mathbb{N}} : a_i \in \mathbb{R} \text{ and } \sum_{i \in \mathbb{N}} a_i^2 < \infty\},$$

the affine subspaces

$$X_n = \{(a_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{N}) : a_0 = a_1 = \cdots = a_n = 1\}$$

with $u_n \colon X_{n+1} \to X_n$ the inclusion map. Then we have

$$\varprojlim (X_n, u_n) = \bigcap_{n \in \mathbb{N}} X_n = \varnothing.$$

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This shows that $\Delta_G(\mathbb{R}^V)$ is closed in $\mathbb{R}^V$ for the prodiscrete topology.

Concluding remarks

Remark

An analogous proof yields the surjectivity of $L = \Delta_G + \lambda \operatorname{Id} \colon \mathbb{R}^V \to \mathbb{R}^V$ for every connected infinite and locally finite simplicial graph G and any real-valued function $\lambda \colon V \to [0, +\infty)$ defined on the vertex set of G. Indeed, L is linear and, for $f \in \mathbb{R}^V$ and $v \in V$, the equality $L(f)(v) = 0$ implies

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\begin{aligned}
|f(v)| &= \left| \frac{1}{(1+\lambda(v)) \deg(v)} \sum_{v \sim w} f(w) \right| \\
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which shows that L also satisfies the maximum principle.

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Remark

In [CC-2009], the surjectivity of Δ_G had been established in the particular case when G is the Cayley graph of a finitely generated infinite group. The proof distinguished two cases according to whether G is amenable or not, by using the Garden of Eden theorem for linear cellular automata [CC-2006] in the amenable case and the Kesten-Day spectral theorem in the non-amenable case.

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