Some properties of injective cellular automata over algebraic varieties

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This is joint work with Tullio Ceccherini-Silberstein and Xuan Kien Phung

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[CCP-2017] T. Ceccherini-Silberstein, M. Coornaert, X.K. Phung, *On injective cellular automata over schemes*, arXiv:1712.05716

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The shift on A^G is the left action of G on A^G given by

$$\begin{aligned} G \times A^G &\to A^G \\ (g, x) &\mapsto gx \end{aligned}$$

where

$$gx(h) := x(g^{-1}h) \quad \forall h \in G.$$

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satisfying the following condition:
there exist a finite subset $M \subset G$ and a map $\mu \colon A^M \to A$ such that

$$(\tau(x))(g) = \mu((g^{-1}x)|_M) \quad \forall x \in A^G, \forall g \in G,$$

where $(g^{-1}x)|_M$ denotes the restriction of the configuration $g^{-1}x$ to M.

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Such a set M is called a memory set and μ is called a local defining map for τ.

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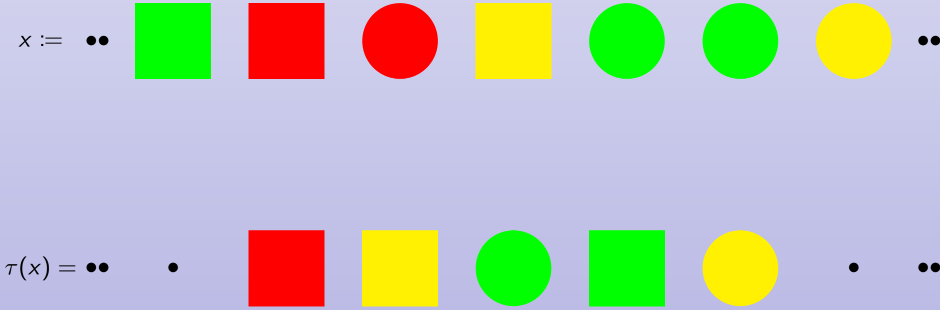
is a cellular automaton with memory set $M = \{-1, 1\}$.

Example: a one-dimensional cellular automaton (continued)



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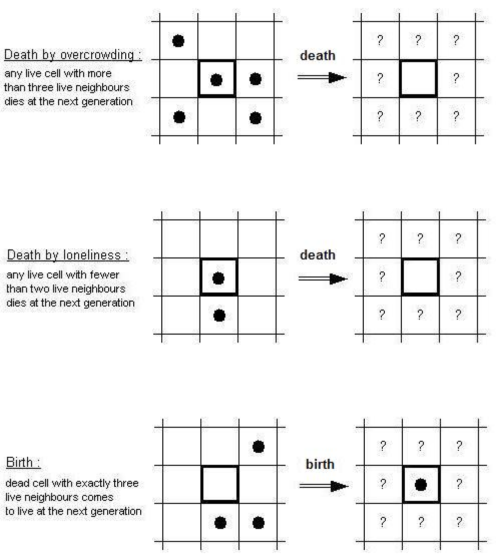

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with memory set $M := \{-1, 0, 1\}^2 \subset \mathbb{Z}^2$
and local defining map $\mu \colon A^M \to A$ given by

$$\mu(y) := \begin{cases} 1 & \text{if} \begin{cases} \displaystyle\sum_{m \in M} y(m) = 3 \\ \text{or} \displaystyle\sum_{m \in M} y(m) = 4 \text{ and } y((0,0)) = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

$\forall y \in A^M$.

Algebraic subsets

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Let K be a field and let m, n be non-negative integers.

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Definition

A subset $A \subset K^m$ is called an algebraic subset if there exists a subset $S \subset K[t_1, \dots, t_m]$ such that A is the set of common zeroes of the polynomials in S, i.e.,

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Definition

Let $A \subset K^m$ and $B \subset K^n$ be algebraic subsets.
A map $f \colon A \to B$ is called regular if f is the restriction of some polynomial map $P \colon K^m \to K^n$.

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Thus, the algebraic subsets of K^m, $m = 0, 1, \ldots$, are the objects of a category whose morphisms are the regular maps.
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This category admits finite products. Indeed, if $A \subset K^m$ and $B \subset K^n$ are algebraic subsets then

$$A \times B \subset K^m \times K^n = K^{m+n}$$

is also an algebraic subset. It is the direct product of A and B in the category of algebraic sets over K.

Algebraic cellular automata

Definition

Let G be a group and let K be a field. One says that a cellular automaton $\tau \colon A^G \to A^G$ with memory set M is an algebraic cellular automaton over K if:
A is an affine algebraic set over K;
the associated local defining map $\mu_M \colon A^M \to A$ is regular.

Examples of algebraic cellular automata

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The map $\tau\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ defined by

$$\tau(x)(n) = x(n+1) - x(n)^2 \quad \forall x \in K^{\mathbb{Z}}, \forall n \in \mathbb{Z},$$

is an algebraic cellular automaton with memory set $M = \{0, 1\}$.

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Remark

Every cellular automaton with finite alphabet A may be regarded as an algebraic cellular automaton (embed A in some field K).

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The category of S-schemes admits finite products (S-fibered products).

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If $f \colon X \to Z$ is an S-scheme morphism, then f induces by precomposition a map

$$f^{(Y)} \colon X(Y) \to Z(Y).$$

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Definition (CCP-2017)

A cellular automaton over the group G and the S-scheme X with coefficients in the S-scheme Y is a cellular automaton $\tau \colon A^G \to A^G$ that admits a memory set M such that the associated local defining map $\mu \colon A^M \to A$ satisfies $\mu = f^{(Y)}$ for some S-scheme morphism $f \colon X^M \to X$.

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Here X^M denotes the S-fibered product of a family of copies of X indexed by M.
The above definition makes sense since $f^{(Y)} \colon (X^M)(Y) \to X(Y) = A$ and $(X^M)(Y) = (X(Y))^M = A^M$ by the universal property of S-fibered products.

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Take $S = Y := K$, where, by a common abuse of notation, K stands for the spectrum of K.
Let $X := \mathrm{Spec}(R)$, where R is the coordinate ring of A, that is, $R := K[t_1, \dots, t_m]/I$, where $I \subset K[t_1, \dots, t_m]$ is the ideal consisting of the polynomials that are identically 0 on A.

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Then $X(Y) = X(K) = A$ and regular maps $A^M \to A$ are the maps that are induced by K-scheme morphisms $X^M \to X$.
Thus, the algebraic cellular automata $\tau \colon A^G \to A^G$ are precisely the cellular automata over the K-scheme X with coefficients in K.

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This means that X is a finite union of affine open subschemes $U_i = \mathrm{Spec}(R_i)$, where each R_i is a finitely generated K-algebra.

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Theorem (CCP-2017, Theorem 1.2)

Suppose that G is locally residually finite. Suppose that the algebraic variety X is complete or that the field K is uncountable. Then

$$\tau \text{ injective} \implies \tau \text{ surjective.}$$

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The cellular automata over a group G and schemes S, X, Y form a monoid for the composition of maps. This monoid is denoted by $\mathrm{CA}(G, S, X, Y)$.

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Theorem (CCP-2017, Theorem 1.4)

Suppose that G is locally residually finite. Suppose that the algebraic variety X is separated and reduced and that the field K has characteristic 0. Then

$$\tau \text{ reversible} \implies \tau \text{ invertible in the monoid } \mathrm{CA}(G, K, X, K).$$

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(Count-2) Take $B := K[t] \subset A$ and consider the restriction $\sigma \colon B^{\mathbb{Z}} \to B^{\mathbb{Z}}$ of τ to $B^{\mathbb{Z}} \subset A^{\mathbb{Z}}$.

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(Count-2) Take $B := K[t] \subset A$ and consider the restriction $\sigma\colon B^{\mathbb{Z}} \to B^{\mathbb{Z}}$ of τ to $B^{\mathbb{Z}} \subset A^{\mathbb{Z}}$. Then σ is an injective cellular automaton that is not surjective.

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