The Garden of Eden Theorem: from Conway's Game of Life to Arnold's Cat

Michel Coornaert

IRMA, Université de Strasbourg

Galatasaray University, Istanbul

This is joint work with Tullio Ceccherini-Silberstein.

Symbolic Dynamics

Symbolic Dynamics

Take:

- a group G (called the universe),

Symbolic Dynamics

Take:

- a group G (called the universe),
- a finite set A (called the alphabet).

Symbolic Dynamics

Take:

- a group G (called the universe),
- a finite set A (called the alphabet).

The set

$$A^G = \{x \colon G \to A\}$$

is called the set of configurations.

Take:

- a group G (called the universe),
- a finite set A (called the alphabet).

The set

$$A^G = \{x \colon G \to A\}$$

is called the set of configurations.
The shift on A^G is the left action of G on A^G given by

$$\begin{aligned} G \times A^G &\to A^G \\ (g, x) &\mapsto gx \end{aligned}$$

where

$$gx(h) = x(g^{-1}h) \quad \forall h \in G.$$

Cellular Automata

Cellular Automata

Definition

A cellular automaton over the group G and the alphabet A is a map

$$\tau\colon A^G \to A^G$$

Cellular Automata

Definition

A cellular automaton over the group G and the alphabet A is a map

$$\tau \colon A^G \to A^G$$

satisfying the following condition:
there exist a finite subset $M \subset G$ and a map $\mu \colon A^M \to A$ such that

$$(\tau(x))(g) = \mu((g^{-1}x)|_M) \quad \forall x \in A^G, \forall g \in G,$$

where $(g^{-1}x)|_M$ denotes the restriction of the configuration $g^{-1}x$ to M.

Definition

A cellular automaton over the group G and the alphabet A is a map

$$\tau \colon A^G \to A^G$$

satisfying the following condition:
there exist a finite subset $M \subset G$ and a map $\mu \colon A^M \to A$ such that

$$(\tau(x))(g) = \mu((g^{-1}x)|_M) \quad \forall x \in A^G, \forall g \in G,$$

where $(g^{-1}x)|_M$ denotes the restriction of the configuration $g^{-1}x$ to M.

Such a set M is called a memory set and μ is called a local defining map for τ.

Example 1

Example 1

Take $G \coloneqq \mathbb{Z}$, $A \coloneqq \{0,1\}$, and $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by

Example 1

Take $G := \mathbb{Z}$, $A := \{0,1\}$, and $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by

$$\tau(x)(n) := x(n) + x(n+1) \mod 2 \quad \forall x \in A^{\mathbb{Z}}, n \in \mathbb{Z}.$$

Example 1

Take $G := \mathbb{Z}$, $A := \{0,1\}$, and $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by

$$\tau(x)(n) := x(n) + x(n+1) \quad \text{mod } 2 \quad \forall x \in A^{\mathbb{Z}}, n \in \mathbb{Z}.$$

For example, if

$$x = \ldots 11100101001101110000100011 \ldots \text{ then}$$
$$\tau(x) = \ldots 0010111101011001000110010? \ldots$$

Example 1

Take $G := \mathbb{Z}$, $A := \{0,1\}$, and $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by

$$\tau(x)(n) := x(n) + x(n+1) \quad \text{mod } 2 \quad \forall x \in A^{\mathbb{Z}}, n \in \mathbb{Z}.$$

For example, if

$$x = \dots 1110010100110111000010001 1\dots \text{ then}$$
$$\tau(x) = \dots 0010111101011001000110010?\dots$$

τ is a cellular automaton

Example 1

Take $G := \mathbb{Z}$, $A := \{0, 1\}$, and $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by

$$\tau(x)(n) := x(n) + x(n+1) \mod 2 \quad \forall x \in A^{\mathbb{Z}}, n \in \mathbb{Z}.$$

For example, if

$$x = \ldots 11100101001101110000100011 \ldots \text{ then}$$
$$\tau(x) = \ldots 0010111101011001000110010? \ldots$$

τ is a cellular automaton with memory $M = \{0, 1\}$ and local defining map $\mu\colon A^M = A^2 \to A$ given by

$$00 \mapsto 0, \quad 01 \mapsto 1, \quad 10 \mapsto 1, \quad 11 \mapsto 0.$$

Example 2

Example 2

Take $G := \mathbb{Z}$ and $A := \{(\text{shape}, \text{color})\}$, where

Example 2

Take $G := \mathbb{Z}$ and $A := \{(\text{shape}, \text{color})\}$, where

$$\text{shape} \in \{\text{square}, \text{disc}\} \quad \text{color} \in \{\text{green}, \text{red}, \text{yellow}\}.$$

Example 2

Take $G := \mathbb{Z}$ and $A := \{(\text{shape}, \text{color})\}$, where

$$\text{shape} \in \{\text{square}, \text{disc}\} \quad \text{color} \in \{\text{green}, \text{red}, \text{yellow}\}.$$

Thus the alphabet A has cardinality $2 \times 3 = 6$.

Example 2

Take $G := \mathbb{Z}$ and $A := \{(\text{shape}, \text{color})\}$, where

$$\text{shape} \in \{\text{square}, \text{disc}\} \quad \text{color} \in \{\text{green}, \text{red}, \text{yellow}\}.$$

Thus the alphabet A has cardinality $2 \times 3 = 6$. The map $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined by

$$\text{shape}((\tau(x))(n)) = \text{shape}(x(n-1)) \quad \text{color}((\tau(x))(n)) = \text{color}(x(n+1)) \quad \forall x \in A^{\mathbb{Z}}, n \in \mathbb{Z},$$

Example 2

Take $G := \mathbb{Z}$ and $A := \{(\text{shape}, \text{color})\}$, where

$$\text{shape} \in \{\text{square}, \text{disc}\} \quad \text{color} \in \{\text{green}, \text{red}, \text{yellow}\}.$$

Thus the alphabet A has cardinality $2 \times 3 = 6$. The map $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined by

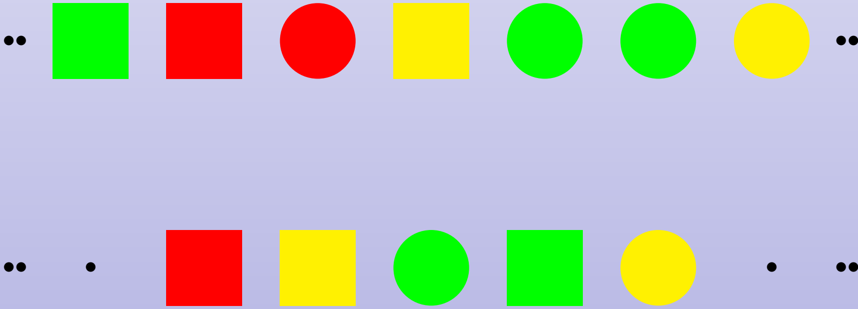
$$\text{shape}((\tau(x))(n)) = \text{shape}(x(n-1)) \quad \text{color}((\tau(x))(n)) = \text{color}(x(n+1)) \quad \forall x \in A^{\mathbb{Z}}, n \in \mathbb{Z},$$

is a cellular automaton with memory set $M = \{-1, 1\}$.

Example 2 (continued)

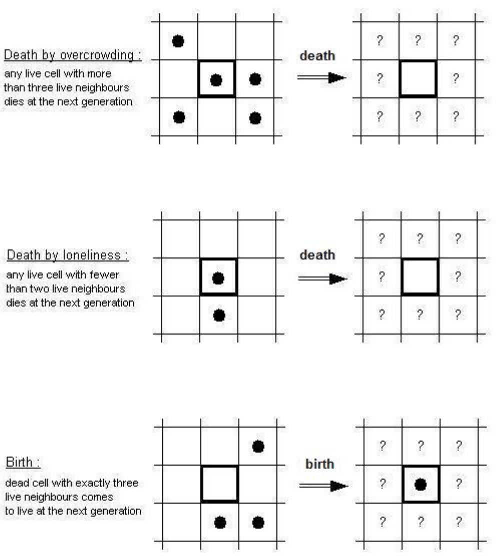
$x :=$

$\tau(x) =$



Example 3: Conway's Game of Life

Example 3: Conway's Game of Life



Example 3 (continued)

Example 3 (continued)

Here we take $G := \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $A := \{0, 1\}$.

Example 3 (continued)

Here we take $G := \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $A := \{0, 1\}$.
Life is described by the cellular automaton

$$\tau \colon A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$$

Example 3 (continued)

Here we take $G := \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $A := \{0, 1\}$.
Life is described by the cellular automaton

$$\tau \colon A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$$

with memory set $M = \{-1, 0, 1\}^2 \subset \mathbb{Z}^2$
and local defining map $\mu \colon A^M \to A$ given by

$$\mu(y) = \begin{cases} 1 & \text{if} \begin{cases} \sum_{m \in M} y(m) = 3 \\ \text{or} \sum_{m \in M} y(m) = 4 \text{ and } y((0,0)) = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

$\forall y \in A^M$.

Diamonds and Pre-injectivity

Diamonds and Pre-injectivity

Let $\tau\colon A^G \to A^G$ be a cellular automaton.

Let $\tau\colon A^G \to A^G$ be a cellular automaton.

Definition

Two configurations $x_1, x_2 \in A^G$ are almost equal if they coincide outside of a finite subset of G, i.e., the set $\{g \in G : x_1(g) \neq x_2(g)\}$ is finite.

Let $\tau \colon A^G \to A^G$ be a cellular automaton.

Definition

Two configurations $x_1, x_2 \in A^G$ are almost equal if they coincide outside of a finite subset of G, i.e., the set $\{g \in G : x_1(g) \neq x_2(g)\}$ is finite.

Definition

Two configurations $x_1, x_2 \in A^G$ form a diamond for τ if

- x_1 and x_2 are almost equal but not equal;
- $\tau(x_1) = \tau(x_2)$.

Let $\tau \colon A^G \to A^G$ be a cellular automaton.

Definition

Two configurations $x_1, x_2 \in A^G$ are almost equal if they coincide outside of a finite subset of G, i.e., the set $\{g \in G : x_1(g) \neq x_2(g)\}$ is finite.

Definition

Two configurations $x_1, x_2 \in A^G$ form a diamond for τ if

- x_1 and x_2 are almost equal but not equal;
- $\tau(x_1) = \tau(x_2)$.

Definition

One says that τ is pre-injective if it has no diamonds.

Diamonds and Pre-injectivity (continued)

Diamonds and Pre-injectivity (continued)

Note that

$$\tau \text{ injective} \implies \tau \text{ pre-injective.}$$

Diamonds and Pre-injectivity (continued)

Note that

$$\tau \text{ injective} \implies \tau \text{ pre-injective.}$$

The converse is false.

Diamonds and Pre-injectivity (continued)

Note that

$$\tau \text{ injective} \implies \tau \text{ pre-injective.}$$

The converse is false.

- Example 1 is pre-injective but not injective;

Diamonds and Pre-injectivity (continued)

Note that

$$\tau \text{ injective } \Longrightarrow \tau \text{ pre-injective.}$$

The converse is false.

- Example 1 is pre-injective but not injective;
- Example 2 is injective and hence pre-injective;

Diamonds and Pre-injectivity (continued)

Note that

$$\tau \text{ injective} \implies \tau \text{ pre-injective.}$$

The converse is false.

- Example 1 is pre-injective but not injective;
- Example 2 is injective and hence pre-injective;
- Example 3 is not pre-injective and hence non-injective;

The GOE Theorem for $\mathbb{Z}^d$

The GOE Theorem for $\mathbb{Z}^d$

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

The GOE Theorem for $\mathbb{Z}^d$

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

Theorem (GOE theorem)

Let $G = \mathbb{Z}^d$ and A a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton. Then

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

The GOE Theorem for $\mathbb{Z}^d$

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

Theorem (GOE theorem)

Let $G = \mathbb{Z}^d$ and A a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton. Then

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

- $\Longrightarrow$ is due to Moore,

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

Theorem (GOE theorem)

Let $G = \mathbb{Z}^d$ and A a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton. Then

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

- $\Longrightarrow$ is due to Moore,
- $\Longleftarrow$ is due to Myhill.

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

Theorem (GOE theorem)

Let $G = \mathbb{Z}^d$ and A a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton. Then

$$\tau \text{ surjective } \iff \tau \text{ pre-injective.}$$

- $\Longrightarrow$ is due to Moore,
- $\Longleftarrow$ is due to Myhill.

- Examples 1 and 2 are pre-injective.

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

Theorem (GOE theorem)

Let $G = \mathbb{Z}^d$ and A a finite set. Let $\tau\colon A^G \to A^G$ be a cellular automaton. Then

$$\tau \text{ surjective } \iff \tau \text{ pre-injective.}$$

- $\implies$ is due to Moore,
- $\impliedby$ is due to Myhill.

- Examples 1 and 2 are pre-injective. Therefore they are surjective (easy to check directly) by Myhill's implication.

The GOE Theorem for $\mathbb{Z}^d$

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

Theorem (GOE theorem)

Let $G = \mathbb{Z}^d$ and A a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton. Then

$$\tau \text{ surjective } \iff \tau \text{ pre-injective.}$$

- $\implies$ is due to Moore,
- $\impliedby$ is due to Myhill.

- Examples 1 and 2 are pre-injective. Therefore they are surjective (easy to check directly) by Myhill's implication.
- Example 3 is not pre-injective.

The GOE Theorem for $\mathbb{Z}^d$

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

Theorem (GOE theorem)

Let $G = \mathbb{Z}^d$ and A a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton. Then

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

- $\Longrightarrow$ is due to Moore,
- $\Longleftarrow$ is due to Myhill.

- Examples 1 and 2 are pre-injective. Therefore they are surjective (easy to check directly) by Myhill's implication.
- Example 3 is not pre-injective.
  Therefore it is not surjective (not easy to check directly) by Moore's implication.

The GOE theorem for Groups of Subexponential Growth

The GOE theorem for Groups of Subexponential Growth

Schupp [S-1988] asked the following.

The GOE theorem for Groups of Subexponential Growth

Schupp [S-1988] asked the following.

Question

Is the analogue of the Moore-Myhill theorem true exactly for virtually nilpotent groups?

The GOE theorem for Groups of Subexponential Growth

Schupp [S-1988] asked the following.

Question

Is the analogue of the Moore-Myhill theorem true exactly for virtually nilpotent groups?

Definition

A group G with finite generating set S has subexponential growth if

$$\lim_{k \to \infty} \frac{\log \mathrm{N}_S(k)}{k} = 0,$$

where $\mathrm{N}_S(k)$ is the number of elements of G that can be written as a product of at most k elements in $S \cup S^{-1}$.

The GOE theorem for Groups of Subexponential Growth

Schupp [S-1988] asked the following.

Question

Is the analogue of the Moore-Myhill theorem true exactly for virtually nilpotent groups?

Definition

A group G with finite generating set S has subexponential growth if

$$\lim_{k \to \infty} \frac{\log \mathrm{N}_S(k)}{k} = 0,$$

where $\mathrm{N}_S(k)$ is the number of elements of G that can be written as a product of at most k elements in $S \cup S^{-1}$.

Machì and Mignosi [MM-1993] proved that the GOE theorem remains valid when G is a f.g. group with subexponential growth.

The GOE theorem for Groups of Subexponential Growth

Schupp [S-1988] asked the following.

Question

Is the analogue of the Moore-Myhill theorem true exactly for virtually nilpotent groups?

Definition

A group G with finite generating set S has subexponential growth if

$$\lim_{k \to \infty} \frac{\log \mathrm{N}_S(k)}{k} = 0,$$

where $\mathrm{N}_S(k)$ is the number of elements of G that can be written as a product of at most k elements in $S \cup S^{-1}$.

Machì and Mignosi [MM-1993] proved that the GOE theorem remains valid when G is a f.g. group with subexponential growth.
Every f.g. virtually nilpotent group has subexponential growth but there are f.g. groups of subexponential growth that are not virtually nilpotent.

The GOE theorem for Groups of Subexponential Growth

Schupp [S-1988] asked the following.

Question

Is the analogue of the Moore-Myhill theorem true exactly for virtually nilpotent groups?

Definition

A group G with finite generating set S has subexponential growth if

$$\lim_{k\to\infty} \frac{\log \mathrm{N}_S(k)}{k} = 0,$$

where $\mathrm{N}_S(k)$ is the number of elements of G that can be written as a product of at most k elements in $S \cup S^{-1}$.

Machì and Mignosi [MM-1993] proved that the GOE theorem remains valid when G is a f.g. group with subexponential growth.
Every f.g. virtually nilpotent group has subexponential growth but there are f.g. groups of subexponential growth that are not virtually nilpotent. The first examples of such groups were given by Grigorchuk [Gri-1984].

The GOE Theorem for Amenable Groups

The GOE Theorem for Amenable Groups

Definition

A group G is amenable if there exists a finitely-additive invariant probability measure defined on the set of all subsets of G.

Definition

A group G is amenable if there exists a finitely-additive invariant probability measure defined on the set of all subsets of G.

All f.g. groups of subexponential growth, all solvable groups, all locally finite groups are amenable.

Definition

A group G is amenable if there exists a finitely-additive invariant probability measure defined on the set of all subsets of G.

All f.g. groups of subexponential growth, all solvable groups, all locally finite groups are amenable.
Ceccherini-Silberstein, Machì and Scarabotti [CMS-1999] proved that the GOE theorem remains valid for amenable groups.

The GOE Theorem for Amenable Groups

Definition

A group G is amenable if there exists a finitely-additive invariant probability measure defined on the set of all subsets of G.

All f.g. groups of subexponential growth, all solvable groups, all locally finite groups are amenable.

Ceccherini-Silberstein, Machì and Scarabotti [CMS-1999] proved that the GOE theorem remains valid for amenable groups.

Bartholdi [B-2010] proved that if G is a non-amenable group then G does not satisfy Moore's implication, i.e., there exist a finite set A and a cellular automaton $\tau \colon A^G \to A^G$ that is surjective but not pre-injective.

The GOE Theorem for Amenable Groups

Definition

A group G is amenable if there exists a finitely-additive invariant probability measure defined on the set of all subsets of G.

All f.g. groups of subexponential growth, all solvable groups, all locally finite groups are amenable.
Ceccherini-Silberstein, Machì and Scarabotti [CMS-1999] proved that the GOE theorem remains valid for amenable groups.
Bartholdi [B-2010] proved that if G is a non-amenable group then G does not satisfy Moore's implication, i.e., there exist a finite set A and a cellular automaton $\tau\colon A^G \to A^G$ that is surjective but not pre-injective.
Bartholdi and Kielak [BK-2016] proved that if G is a non-amenable group then G does not satisfy Myhill's implication either, i.e., there exist a finite set A and a cellular automaton $\tau\colon A^G \to A^G$ that is pre-injective but not surjective.

What Gromov Said

What Gromov Said

Gromov [Gro-1999, p. 195] wrote:

What Gromov Said

Gromov [Gro-1999, p. 195] wrote:

> *"...the Garden of Eden theorem can be generalized to a suitable class of hyperbolic actions ..."*

Dynamical systems

Dynamical systems

A dynamical system is a pair (X, G), where

- X is a compact metrizable topological space,

Dynamical systems

A dynamical system is a pair (X, G), where

- X is a compact metrizable topological space,
- G is a countable group acting continuously on X.

Dynamical systems

A dynamical system is a pair (X, G), where

- X is a compact metrizable topological space,
- G is a countable group acting continuously on X.

The space X is called the phase space.

Dynamical systems

A dynamical system is a pair (X, G), where

- X is a compact metrizable topological space,
- G is a countable group acting continuously on X.

The space X is called the phase space.
If $f\colon X \to X$ is a homeomorphism, the d.s. generated by f is the d.s. $(X, \mathbb{Z})$, where $\mathbb{Z}$ acts on X by

$$(n, x) \mapsto f^n(x) \quad \forall n \in \mathbb{Z}, x \in X.$$

This d.s. is also denoted (X, f).

Examples of Dynamical Systems

Example

Let A be a finite set and G a countable group.

Example

Let A be a finite set and G a countable group. Equip A with its discree topology and A^G with the product topology.

Examples of Dynamical Systems

Example

Let A be a finite set and G a countable group. Equip A with its discree topology and A^G with the product topology. Then the shift (A^G, G) is a d.s.

Examples of Dynamical Systems

Example

Let A be a finite set and G a countable group. Equip A with its discree topology and A^G with the product topology. Then the shift (A^G, G) is a d.s.

Example (Arnold's cat)

This is the d.s. $(\mathbb{T}^2, f)$, where f is the automorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ given by

$$f(x) = \begin{pmatrix} x_2 \\ x_1 + x_2 \end{pmatrix} \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{T}^2.$$

Example

Let A be a finite set and G a countable group. Equip A with its discree topology and A^G with the product topology. Then the shift (A^G, G) is a d.s.

Example (Arnold's cat)

This is the d.s. $(\mathbb{T}^2, f)$, where f is the automorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ given by

$$f(x) = \begin{pmatrix} x_2 \\ x_1 + x_2 \end{pmatrix} \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{T}^2.$$

Thus we have $f(x) = Ax$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is the cat matrix.

Homoclinicity

Homoclinicity

Let (X, G) be a dynamical system.

Homoclinicity

Let (X, G) be a dynamical system. Let d be a metric on X that is compatible with the topology.

Homoclinicity

Let (X, G) be a dynamical system. Let d be a metric on X that is compatible with the topology.

Definition

Two points $x, y \in X$ are caled homoclinic if

$$\lim_{g \to \infty} d(gx, gy) = 0,$$

Let (X, G) be a dynamical system. Let d be a metric on X that is compatible with the topology.

Definition

Two points $x, y \in X$ are caled **homoclinic** if

$$\lim_{g \to \infty} d(gx, gy) = 0,$$

i.e., for every $\varepsilon > 0$, there exists a finite subset $F \subset G$ such that

$$d(gx, gy) < \varepsilon \quad \forall g \in G \setminus F.$$

Homoclinicity

Let (X, G) be a dynamical system. Let d be a metric on X that is compatible with the topology.

Definition

Two points $x, y \in X$ are caled homoclinic if

$$\lim_{g \to \infty} d(gx, gy) = 0,$$

i.e., for every $\varepsilon > 0$, there exists a finite subset $F \subset G$ such that

$$d(gx, gy) < \varepsilon \quad \forall g \in G \setminus F.$$

Homoclinicity is an equivalence relation on X.

Homoclinicity

Let (X, G) be a dynamical system. Let d be a metric on X that is compatible with the topology.

Definition

Two points $x, y \in X$ are caled homoclinic if

$$\lim_{g \to \infty} d(gx, gy) = 0,$$

i.e., for every $\varepsilon > 0$, there exists a finite subset $F \subset G$ such that

$$d(gx, gy) < \varepsilon \quad \forall g \in G \setminus F.$$

Homoclinicity is an equivalence relation on X. This relation does not depend on the choice of d.

Homoclinicity (continued)

Homoclinicity (continued)

Example

Let A be a finite set and G a countable group.

Homoclinicity (continued)

Example

Let A be a finite set and G a countable group. Consider the shift (A^G, G).

Homoclinicity (continued)

Example

Let A be a finite set and G a countable group. Consider the shift (A^G, G).
Two configurations $x, y \in A^G$ are homoclinic if and only if they are almost equal.

Homoclinicity (continued)

Example

Let A be a finite set and G a countable group. Consider the shift (A^G, G).
Two configurations $x, y \in A^G$ are homoclinic if and only if they are almost equal.

Example

Consider Arnold's cat $(\mathbb{T}^2, f)$.

Homoclinicity (continued)

Example

Let A be a finite set and G a countable group. Consider the shift (A^G, G).
Two configurations $x, y \in A^G$ are homoclinic if and only if they are almost equal.

Example

Consider Arnold's cat $(\mathbb{T}^2, f)$.
Equip $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with its Euclidean structure.

Homoclinicity (continued)

Example

Let A be a finite set and G a countable group. Consider the shift (A^G, G).
Two configurations $x, y \in A^G$ are homoclinic if and only if they are almost equal.

Example

Consider Arnold's cat $(\mathbb{T}^2, f)$.
Equip $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with its Euclidean structure.
The homoclinicity class of a point $x \in \mathbb{T}^2$ is $D \cap D'$, where D is the line passing through x whose slope is the golden mean $\dfrac{1+\sqrt{5}}{2} = 1.618\ldots$ and D' is the line passing through x and orthogonal to D'.

Homoclinicity (continued)

Example

Let A be a finite set and G a countable group. Consider the shift (A^G, G).
Two configurations $x, y \in A^G$ are homoclinic if and only if they are almost equal.

Example

Consider Arnold's cat $(\mathbb{T}^2, f)$.
Equip $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with its Euclidean structure.
The homoclinicity class of a point $x \in \mathbb{T}^2$ is $D \cap D'$, where D is the line passing through x whose slope is the golden mean $\dfrac{1+\sqrt{5}}{2} = 1.618\ldots$ and D' is the line passing through x and orthogonal to D'. The slopes of D and D' are the eigenvalues of the cat matrix.

Homoclinicity (continued)

Example

Let A be a finite set and G a countable group. Consider the shift (A^G, G).
Two configurations $x, y \in A^G$ are homoclinic if and only if they are almost equal.

Example

Consider Arnold's cat $(\mathbb{T}^2, f)$.
Equip $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with its Euclidean structure.
The homoclinicity class of a point $x \in \mathbb{T}^2$ is $D \cap D'$, where D is the line passing through x whose slope is the golden mean $\dfrac{1+\sqrt{5}}{2} = 1.618\ldots$ and D' is the line passing through x and orthogonal to D'. The slopes of D and D' are the eigenvalues of the cat matrix.
Each homoclinicity class is countably-infinite and dense in $\mathbb{T}^2$.

Endomorphisms of Dynamical Systems

Endomorphisms of Dynamical Systems

Let (X, G) be a dynamical system.

Let (X, G) be a dynamical system.

Definition

An endomorphism of the d.s. (X, G) is a continuous map $\tau \colon X \to X$ such that τ commutes with the action of G, that is, such that

$$\tau(gx) = g\tau(x) \quad \forall g \in G, x \in X.$$

Let (X, G) be a dynamical system.

Definition

An endomorphism of the d.s. (X, G) is a continuous map $\tau \colon X \to X$ such that τ commutes with the action of G, that is, such that

$$\tau(gx) = g\tau(x) \quad \forall g \in G, x \in X.$$

Example

Let A be a finite set and G a countable group.

Endomorphisms of Dynamical Systems

Let (X, G) be a dynamical system.

Definition

An endomorphism of the d.s. (X, G) is a continuous map $\tau \colon X \to X$ such that τ commutes with the action of G, that is, such that

$$\tau(gx) = g\tau(x) \quad \forall g \in G, x \in X.$$

Example

Let A be a finite set and G a countable group.
Then the endomorphisms of the shift (A^G, G) are precisely the cellular automata $\tau \colon A^G \to A^G$

Let (X, G) be a dynamical system.

Definition

An endomorphism of the d.s. (X, G) is a continuous map $\tau\colon X \to X$ such that τ commutes with the action of G, that is, such that

$$\tau(gx) = g\tau(x) \quad \forall g \in G, x \in X.$$

Example

Let A be a finite set and G a countable group.
Then the endomorphisms of the shift (A^G, G) are precisely the cellular automata $\tau\colon A^G \to A^G$
(Curtis-Hedlund-Lyndon theorem).

Pre-injective Endomorphisms

Pre-injective Endomorphisms

Let (X, G) be a dynamical system.

Pre-injective Endomorphisms

Let (X, G) be a dynamical system.

Definition

An endomorphism $\tau \colon X \to X$ of the d.s. (X, G) is called pre-injective if its restriction to each homoclinicity class is injective.

Pre-injective Endomorphisms

Let (X, G) be a dynamical system.

Definition

An endomorphism $\tau \colon X \to X$ of the d.s. (X, G) is called pre-injective if its restriction to each homoclinicity class is injective.

Example

For shift systems (A^G, G), the two definitions of pre-injectivity are equivalent.

Pre-injective Endomorphisms

Let (X, G) be a dynamical system.

Definition

An endomorphism $\tau \colon X \to X$ of the d.s. (X, G) is called pre-injective if its restriction to each homoclinicity class is injective.

Example

For shift systems (A^G, G), the two definitions of pre-injectivity are equivalent.

Example

The group endomorphism $\tau \colon \mathbb{T}^2 \to \mathbb{T}^2$, given by $\tau(x) := 2x$ for all $x \in \mathbb{T}^2$, is an endomorphism of Arnold's cat $(\mathbb{T}^2, f)$.

Pre-injective Endomorphisms

Let (X, G) be a dynamical system.

Definition

An endomorphism $\tau \colon X \to X$ of the d.s. (X, G) is called pre-injective if its restriction to each homoclinicity class is injective.

Example

For shift systems (A^G, G), the two definitions of pre-injectivity are equivalent.

Example

The group endomorphism $\tau \colon \mathbb{T}^2 \to \mathbb{T}^2$, given by $\tau(x) := 2x$ for all $x \in \mathbb{T}^2$, is an endomorphism of Arnold's cat $(\mathbb{T}^2, f)$.
The kernel of τ consists of four points:

$$\mathrm{Ker}(\tau) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\}.$$

Pre-injective Endomorphisms

Let (X, G) be a dynamical system.

Definition

An endomorphism $\tau \colon X \to X$ of the d.s. (X, G) is called pre-injective if its restriction to each homoclinicity class is injective.

Example

For shift systems (A^G, G), the two definitions of pre-injectivity are equivalent.

Example

The group endomorphism $\tau \colon \mathbb{T}^2 \to \mathbb{T}^2$, given by $\tau(x) := 2x$ for all $x \in \mathbb{T}^2$, is an endomorphism of Arnold's cat $(\mathbb{T}^2, f)$.
The kernel of τ consists of four points:

$$\mathrm{Ker}(\tau) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\}.$$

The endomorphism τ is pre-injective but not injective.

Dynamical Systems that Satisfy the GOE Theorem

Dynamical Systems that Satisfy the GOE Theorem

Let (X, G) be a dynamical system.

Let (X, G) be a dynamical system.

Definition

One says that the d.s. (X, G) satisfies the Garden of Eden theorem if every endomorphism $\tau \colon X \to X$ of (X, G) satisfies

Dynamical Systems that Satisfy the GOE Theorem

Let (X, G) be a dynamical system.

Definition

One says that the d.s. (X, G) satisfies the Garden of Eden theorem if every endomorphism $\tau \colon X \to X$ of (X, G) satisfies

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

Dynamical Systems that Satisfy the GOE Theorem

Let (X, G) be a dynamical system.

Definition

One says that the d.s. (X, G) satisfies the Garden of Eden theorem if every endomorphism $\tau\colon X \to X$ of (X, G) satisfies

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

Example

Arnold's cat $(\mathbb{T}^2, f)$ satisfies the GOE theorem.

Dynamical Systems that Satisfy the GOE Theorem

Let (X, G) be a dynamical system.

Definition

One says that the d.s. (X, G) satisfies the Garden of Eden theorem if every endomorphism $\tau \colon X \to X$ of (X, G) satisfies

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

Example

Arnold's cat $(\mathbb{T}^2, f)$ satisfies the GOE theorem. Indeed, it is easy to show, using spectral analysis, that any endomorphism τ of the cat is of the form $\tau = m\,\mathrm{Id} + nf$, for some $m, n \in \mathbb{Z}$.

Dynamical Systems that Satisfy the GOE Theorem

Let (X, G) be a dynamical system.

Definition

One says that the d.s. (X, G) satisfies the Garden of Eden theorem if every endomorphism $\tau \colon X \to X$ of (X, G) satisfies

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

Example

Arnold's cat $(\mathbb{T}^2, f)$ satisfies the GOE theorem. Indeed, it is easy to show, using spectral analysis, that any endomorphism τ of the cat is of the form $\tau = m \operatorname{Id} + nf$, for some $m, n \in \mathbb{Z}$. With the exception of the 0-endomorphism, every endomorphism of the cat is both surjective and pre-injective.

Anosov Diffeomorphisms

Anosov Diffeomorphisms

Let $f\colon M \to M$ be a diffeomorphism of a smooth compact manifold M.

Let $f\colon M \to M$ be a diffeomorphism of a smooth compact manifold M.
One says that f is Anosov if the tangent bundle TM of M continuously splits as a direct sum $TM = E_s \oplus E_u$ of two df-invariant subbundles E_s and E_u such that, with respect to some (or equivalently any) Riemannian metric on M, the differential df is exponentially contracting on E_s and exponentially expanding on E_u, i. e., there are constants $C > 0$ and $0 < \lambda < 1$ such that

- $\|df^n(v)\| \leq C\lambda^n \|v\|$,
- $\|df^{-n}(w)\| \leq C\lambda^n \|w\|$

for all $x \in M$, $v \in E_s(x)$, $w \in E_u(x)$, and $n \geq 0$.

Anosov Diffeomorphisms

Let $f: M \to M$ be a diffeomorphism of a smooth compact manifold M.
One says that f is Anosov if the tangent bundle TM of M continuously splits as a direct sum $TM = E_s \oplus E_u$ of two df-invariant subbundles E_s and E_u such that, with respect to some (or equivalently any) Riemannian metric on M, the differential df is exponentially contracting on E_s and exponentially expanding on E_u, i. e., there are constants $C > 0$ and $0 < \lambda < 1$ such that

- $\|df^n(v)\| \leq C\lambda^n \|v\|$,
- $\|df^{-n}(w)\| \leq C\lambda^n \|w\|$

for all $x \in M$, $v \in E_s(x)$, $w \in E_u(x)$, and $n \geq 0$.

Example

Arnold's cat is Anosov.

Anosov Diffeomorphisms

Let $f\colon M \to M$ be a diffeomorphism of a smooth compact manifold M.
One says that f is Anosov if the tangent bundle TM of M continuously splits as a direct sum $TM = E_s \oplus E_u$ of two df-invariant subbundles E_s and E_u such that, with respect to some (or equivalently any) Riemannian metric on M, the differential df is exponentially contracting on E_s and exponentially expanding on E_u, i. e., there are constants $C > 0$ and $0 < \lambda < 1$ such that

- $\|df^n(v)\| \leq C\lambda^n\|v\|$,
- $\|df^{-n}(w)\| \leq C\lambda^n\|w\|$

for all $x \in M$, $v \in E_s(x)$, $w \in E_u(x)$, and $n \geq 0$.

Example

Arnold's cat is Anosov.
If we identify the tangent space at $x \in \mathbb{T}^2$ with $\mathbb{R}^2$, the two eigenlines of the cat matrix yield $E_u(x)$ and $E_s(x)$.

Hyperbolic toral automorphisms

Example

Arnold's cat can be generalized as follows.

Hyperbolic toral automorphisms

Example

Arnold's cat can be generalized as follows.
Consider a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ with no eigenvalue of modulus 1. Then the map

$$\begin{aligned} f \colon \mathbb{T}^n &\to \mathbb{T}^n \\ x &\mapsto Ax \end{aligned}$$

is an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$.

Hyperbolic toral automorphisms

Example

Arnold's cat can be generalized as follows.
Consider a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ with no eigenvalue of modulus 1. Then the map

$$\begin{aligned} f \colon \mathbb{T}^n &\to \mathbb{T}^n \\ x &\mapsto Ax \end{aligned}$$

is an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$.
One says that f is the hyperbolic toral automorphism associated with A.

A GOE Theorem for Anosov Diffeomorphisms on Tori

A GOE Theorem for Anosov Diffeomorphisms on Tori

Theorem (CC-2016)

Let f be an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n$. Then the d.s. $(\mathbb{T}^n, f)$ satisfies the GOE theorem.

Theorem (CC-2016)

Let f be an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n$. Then the d.s. $(\mathbb{T}^n, f)$ satisfies the GOE theorem.

The proof uses two classical results:

Theorem (CC-2016)

Let f be an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n$. Then the d.s. $(\mathbb{T}^n, f)$ satisfies the GOE theorem.

The proof uses two classical results:

Result 1 (Franks [Fra-1970], Manning [Man-1974]) Every Anosov diffeomorphisms of $\mathbb{T}^n$ is topologically conjugate to a hyperbolic toral automorphism.

A GOE Theorem for Anosov Diffeomorphisms on Tori

Theorem (CC-2016)

Let f be an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n$. Then the d.s. $(\mathbb{T}^n, f)$ satisfies the GOE theorem.

The proof uses two classical results:

Result 1 (Franks [Fra-1970], Manning [Man-1974]) Every Anosov diffeomorphisms of $\mathbb{T}^n$ is topologically conjugate to a hyperbolic toral automorphism.

Result 2 (Walters [Wal-1968]) Every endomorphism of a hyperbolic toral automorphism on $\mathbb{T}^n$ is affine, i. e., of the form $x \mapsto Bx + c$, where B is an integral $n \times n$ matrix and $c \in \mathbb{T}^n$.

General Anosov Diffoemorphisms

Question

Let f be an Anosov diffeomorphism of a smooth compact manifold M.

General Anosov Diffoemorphisms

Question

Let f be an Anosov diffeomorphism of a smooth compact manifold M. Does the dynamical system (M, f) satisfy the GOE theorem?

General Anosov Diffoemorphisms

Question

Let f be an Anosov diffeomorphism of a smooth compact manifold M. Does the dynamical system (M, f) satisfy the GOE theorem?

A homeomorphism f of a topological space X is topologically mixing if, given any two non-empty open subsets $U, V \subset X$, one has $U \cap f^n(V) \neq \varnothing$ for all but finitely many $n \in \mathbb{Z}$.

General Anosov Diffoemorphisms

Question

Let f be an Anosov diffeomorphism of a smooth compact manifold M. Does the dynamical system (M, f) satisfy the GOE theorem?

A homeomorphism f of a topological space X is topologically mixing if, given any two non-empty open subsets $U, V \subset X$, one has $U \cap f^n(V) \neq \varnothing$ for all but finitely many $n \in \mathbb{Z}$.

Theorem (CC-2015)

Let f be a topologically mixing Anosov diffeomorphism of a smooth compact manifold M. Then (M, f) has the Myhill property, i.e., every pre-injective continuous map $\tau \colon M \to M$ commuting with f is surjective.

Question

Let f be an Anosov diffeomorphism of a smooth compact manifold M. Does the dynamical system (M, f) satisfy the GOE theorem?

A homeomorphism f of a topological space X is topologically mixing if, given any two non-empty open subsets $U, V \subset X$, one has $U \cap f^n(V) \neq \varnothing$ for all but finitely many $n \in \mathbb{Z}$.

Theorem (CC-2015)

Let f be a topologically mixing Anosov diffeomorphism of a smooth compact manifold M. Then (M, f) has the Myhill property, i.e., every pre-injective continuous map $\tau \colon M \to M$ commuting with f is surjective.

Remark

All known examples of Anosov diffeomorphisms are topologically mixing.

Algebraic Dynamical Systems

Algebraic Dynamical Systems

Definition

An algebraic dynamical system is a d.s. (X, G),

Algebraic Dynamical Systems

Definition

An algebraic dynamical system is a d.s. (X, G), where X is a compact metrizable abelian topological group and G is a countable group acting on X by continuous group automorphisms.

Algebraic Dynamical Systems

Definition

An algebraic dynamical system is a d.s. (X, G), where X is a compact metrizable abelian topological group and G is a countable group acting on X by continuous group automorphisms.

Example

Let G be a countable group and A a c.m.a.t. group.

Algebraic Dynamical Systems

Definition

An algebraic dynamical system is a d.s. (X, G), where X is a compact metrizable abelian topological group and G is a countable group acting on X by continuous group automorphisms.

Example

Let G be a countable group and A a c.m.a.t. group. Then A^G is a c.m.a.t. group.

Algebraic Dynamical Systems

Definition

An algebraic dynamical system is a d.s. (X, G), where X is a compact metrizable abelian topological group and G is a countable group acting on X by continuous group automorphisms.

Example

Let G be a countable group and A a c.m.a.t. group. Then A^G is a c.m.a.t. group. The shift system (A^G, G) is an a.d.s.

Algebraic Dynamical Systems

Definition

An algebraic dynamical system is a d.s. (X, G), where X is a compact metrizable abelian topological group and G is a countable group acting on X by continuous group automorphisms.

Example

Let G be a countable group and A a c.m.a.t. group. Then A^G is a c.m.a.t. group. The shift system (A^G, G) is an a.d.s.

Example

Arnold's cat $(\mathbb{T}^2, \mathbb{Z})$ is an a.d.s.

Principal Algebraic Dynamical Systems

Principal Algebraic Dynamical Systems

Let G be a countable group and denote by $\mathbb{Z}[G]$ its integral group ring.

Principal Algebraic Dynamical Systems

Let G be a countable group and denote by $\mathbb{Z}[G]$ its integral group ring.
If M is a countable left $\mathbb{Z}[G]$-module, then its Pontryagin dual $\widehat{M}$ (the character group of the additive group M) is a c.m.a.t. group.

Principal Algebraic Dynamical Systems

Let G be a countable group and denote by $\mathbb{Z}[G]$ its integral group ring.
If M is a countable left $\mathbb{Z}[G]$-module, then its Pontryagin dual $\widehat{M}$ (the character group of the additive group M) is a c.m.a.t. group. G acts on M and hence (by dualizing) on $\widehat{M}$ by continuous group automorphisms.

Principal Algebraic Dynamical Systems

Let G be a countable group and denote by $\mathbb{Z}[G]$ its integral group ring.
If M is a countable left $\mathbb{Z}[G]$-module, then its Pontryagin dual $\widehat{M}$ (the character group of the additive group M) is a c.m.a.t. group. G acts on M and hence (by dualizing) on $\widehat{M}$ by continuous group automorphisms. $(\widehat{M}, G)$ is an a.d.s.

Principal Algebraic Dynamical Systems

Let G be a countable group and denote by $\mathbb{Z}[G]$ its integral group ring.
If M is a countable left $\mathbb{Z}[G]$-module, then its Pontryagin dual $\widehat{M}$ (the character group of the additive group M) is a c.m.a.t. group. G acts on M and hence (by dualizing) on $\widehat{M}$ by continuous group automorphisms. $(\widehat{M}, G)$ is an a.d.s.
Every a.d.s. can be obtained in this way (see [Sch-1995]).

Principal Algebraic Dynamical Systems

Let G be a countable group and denote by $\mathbb{Z}[G]$ its integral group ring.
If M is a countable left $\mathbb{Z}[G]$-module, then its Pontryagin dual $\widehat{M}$ (the character group of the additive group M) is a c.m.a.t. group. G acts on M and hence (by dualizing) on $\widehat{M}$ by continuous group automorphisms. $(\widehat{M}, G)$ is an a.d.s.
Every a.d.s. can be obtained in this way (see [Sch-1995]).
In the case $M = \mathbb{Z}[G]/(f)$, where $(f) = \mathbb{Z}[G]f$ denotes the left ideal generated by $f \in \mathbb{Z}[G]$, one writes $X_f := \widehat{M}$ and one says that (X_f, G) is the principal a.d.s. associated with f.

Convolution Product

Convolution Product

Let G be a countable group.

Convolution Product

Let G be a countable group.

$$\ell^1(G) := \{\eta \colon G \to \mathbb{R} : \sum_{g \in G} |\eta(g)| < \infty\}.$$

Convolution Product

Let G be a countable group.

$$\ell^1(G) := \{\eta \colon G \to \mathbb{R} : \sum_{g \in G} |\eta(g)| < \infty\}.$$

$$\ell^\infty(G) := \{\xi \colon G \to \mathbb{R} : \sup_{g \in G} |\xi(g)| < \infty\}.$$

Convolution Product

Let G be a countable group.

$$\ell^1(G) \coloneqq \{\eta \colon G \to \mathbb{R} : \sum_{g \in G} |\eta(g)| < \infty\}.$$

$$\ell^\infty(G) \coloneqq \{\xi \colon G \to \mathbb{R} : \sup_{g \in G} |\xi(g)| < \infty\}.$$

$$G \subset \mathbb{Z}[G] \subset \ell^1(G) \subset \ell^\infty(g).$$

For $\eta \in \ell^1(G)$ and $\xi \in \ell^\infty(G)$, define $\eta \star \xi$ by

$$(\eta \star \xi)(g) \coloneqq \sum_{\substack{g_1, g_2 \in G: \\ g_1 g_2 = g}} \eta(g_1)\xi(g_2) = \sum_{h \in G} \eta(gh^{-1})\xi(h).$$

Convolution Product

Let G be a countable group.

$$\ell^1(G) := \{\eta\colon G \to \mathbb{R} : \sum_{g \in G} |\eta(g)| < \infty\}.$$

$$\ell^\infty(G) := \{\xi\colon G \to \mathbb{R} : \sup_{g \in G} |\xi(g)| < \infty\}.$$

$$G \subset \mathbb{Z}[G] \subset \ell^1(G) \subset \ell^\infty(g).$$

For $\eta \in \ell^1(G)$ and $\xi \in \ell^\infty(G)$, define $\eta \star \xi$ by

$$(\eta \star \xi)(g) := \sum_{\substack{g_1, g_2 \in G: \\ g_1 g_2 = g}} \eta(g_1)\xi(g_2) = \sum_{h \in G} \eta(gh^{-1})\xi(h).$$

- $\star$ extends the group operation on G;
- $(\ell^1(G), \star)$ is a Banach algebra.

GOE Theorems for Principal Algebraic Dynamical Systems

Theorem (Li-2017)

Let G be a countable amenable group and let $f \in \mathbb{Z}[G]$ such that f is invertible in $\ell^1(G)$.

Theorem (Li-2017)

Let G be a countable amenable group and let $f \in \mathbb{Z}[G]$ such that f is invertible in $\ell^1(G)$. Then the p.a.d.s. (X_f, G) satisfies the GOE theorem.

Theorem (Li-2017)

Let G be a countable amenable group and let $f \in \mathbb{Z}[G]$ such that f is invertible in $\ell^1(G)$. Then the p.a.d.s. (X_f, G) satisfies the GOE theorem.

The fact that $f \in \mathbb{Z}[G]$ is invertible in $\ell^1(G)$ is equivalent to the expansiveness of (X_f, G).

Theorem (Li-2017)

Let G be a countable amenable group and let $f \in \mathbb{Z}[G]$ such that f is invertible in $\ell^1(G)$. Then the p.a.d.s. (X_f, G) satisfies the GOE theorem.

The fact that $f \in \mathbb{Z}[G]$ is invertible in $\ell^1(G)$ is equivalent to the expansiveness of (X_f, G).
A sufficient condition for $f \in \mathbb{Z}[G]$ to be invertible in $\ell^1(G)$ is that f is lopsided, i.e., there exists $g_0 \in G$ such that

$$|f(g_0)| > \sum_{g \neq g_0} |f(g)|.$$

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

$$\mathcal{C}_0(G) := \{\xi \colon G \to \mathbb{R} : \lim_{g \to \infty} \xi(g) = 0\} \subset \ell^\infty(G).$$

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

$$\mathcal{C}_0(G) := \{\xi \colon G \to \mathbb{R} : \lim_{g \to \infty} \xi(g) = 0\} \subset \ell^\infty(G).$$

Definition

An element $f \in \mathbb{Z}[G]$ is weakly expansive if it satisfies:

(C1) $\forall \xi \in \mathcal{C}_0(G), \quad f \star \xi = 0 \implies \xi = 0;$

(C2) $\exists \omega \in \mathcal{C}_0(G)$ such that $f \star \omega = 1_G$.

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

$$\mathcal{C}_0(G) := \{\xi\colon G \to \mathbb{R} : \lim_{g\to\infty} \xi(g) = 0\} \subset \ell^\infty(G).$$

Definition

An element $f \in \mathbb{Z}[G]$ is weakly expansive if it satisfies:

(C1) $\forall \xi \in \mathcal{C}_0(G), \quad f \star \xi = 0 \implies \xi = 0;$

(C2) $\exists \omega \in \mathcal{C}_0(G)$ such that $f \star \omega = 1_G$.

f invertible in $\ell^1(G) \quad \implies \quad f$ weakly expansive.

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

$$\mathcal{C}_0(G) := \{\xi \colon G \to \mathbb{R} : \lim_{g\to\infty} \xi(g) = 0\} \subset \ell^\infty(G).$$

Definition

An element $f \in \mathbb{Z}[G]$ is weakly expansive if it satisfies:

(C1) $\forall \xi \in \mathcal{C}_0(G), \quad f \star \xi = 0 \implies \xi = 0;$

(C2) $\exists \omega \in \mathcal{C}_0(G)$ such that $f \star \omega = 1_G$.

f invertible in $\ell^1(G) \quad \Longrightarrow \quad f$ weakly expansive.

Theorem (CCL-2018)

Let G be a countable abelian group and let $f \in \mathbb{Z}[G]$ such that f is weakly expansive and X_f is connected.

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

$$\mathcal{C}_0(G) := \{\xi \colon G \to \mathbb{R} : \lim_{g \to \infty} \xi(g) = 0\} \subset \ell^\infty(G).$$

Definition

An element $f \in \mathbb{Z}[G]$ is weakly expansive if it satisfies:

(C1) $\forall \xi \in \mathcal{C}_0(G), \quad f \star \xi = 0 \implies \xi = 0;$

(C2) $\exists \omega \in \mathcal{C}_0(G)$ such that $f \star \omega = 1_G$.

f invertible in $\ell^1(G) \quad \implies \quad f$ weakly expansive.

Theorem (CCL-2018)

Let G be a countable abelian group and let $f \in \mathbb{Z}[G]$ such that f is weakly expansive and X_f is connected. Then the p.a.d.s. (X_f, G) satisfies the GOE theorem.

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

$$\mathcal{C}_0(G) := \{\xi \colon G \to \mathbb{R} : \lim_{g \to \infty} \xi(g) = 0\} \subset \ell^\infty(G).$$

Definition

An element $f \in \mathbb{Z}[G]$ is weakly expansive if it satisfies:

(C1) $\forall \xi \in \mathcal{C}_0(G), \quad f \star \xi = 0 \implies \xi = 0;$

(C2) $\exists \omega \in \mathcal{C}_0(G)$ such that $f \star \omega = 1_G$.

f invertible in $\ell^1(G) \quad \implies \quad f$ weakly expansive.

Theorem (CCL-2018)

Let G be a countable abelian group and let $f \in \mathbb{Z}[G]$ such that f is weakly expansive and X_f is connected. Then the p.a.d.s. (X_f, G) satisfies the GOE theorem.

Example

Take $g := \mathbb{Z}^3$, so that $\mathbb{Z}[G] = \mathbb{Z}[u^\pm, v^\pm, w^\pm]$ is the ring of Laurent polynomials on 3 indeterminates.

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

$$\mathcal{C}_0(G) := \{\xi\colon G \to \mathbb{R} : \lim_{g\to\infty} \xi(g) = 0\} \subset \ell^\infty(G).$$

Definition

An element $f \in \mathbb{Z}[G]$ is weakly expansive if it satisfies:

(C1) $\forall \xi \in \mathcal{C}_0(G), \quad f \star \xi = 0 \implies \xi = 0;$

(C2) $\exists \omega \in \mathcal{C}_0(G)$ such that $f \star \omega = 1_G$.

f invertible in $\ell^1(G) \quad \implies \quad f$ weakly expansive.

Theorem (CCL-2018)

Let G be a countable abelian group and let $f \in \mathbb{Z}[G]$ such that f is weakly expansive and X_f is connected. Then the p.a.d.s. (X_f, G) satisfies the GOE theorem.

Example

Take $g := \mathbb{Z}^3$, so that $\mathbb{Z}[G] = \mathbb{Z}[u^\pm, v^\pm, w^\pm]$ is the ring of Laurent polynomials on 3 indeterminates. Then $f := 6 - u - u^{-1} - v - v^{-1} - w - w^{-1}$ satisfies the hypotheses of the previous theorem but is not invertible in $\ell^1(G)$.

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

$$\mathcal{C}_0(G) := \{\xi\colon G \to \mathbb{R} : \lim_{g\to\infty} \xi(g) = 0\} \subset \ell^\infty(G).$$

Definition

An element $f \in \mathbb{Z}[G]$ is weakly expansive if it satisfies:

(C1) $\forall \xi \in \mathcal{C}_0(G), \quad f \star \xi = 0 \implies \xi = 0;$

(C2) $\exists \omega \in \mathcal{C}_0(G)$ such that $f \star \omega = 1_G$.

f invertible in $\ell^1(G) \quad \Longrightarrow \quad f$ weakly expansive.

Theorem (CCL-2018)

Let G be a countable abelian group and let $f \in \mathbb{Z}[G]$ such that f is weakly expansive and X_f is connected. Then the p.a.d.s. (X_f, G) satisfies the GOE theorem.

Example

Take $g := \mathbb{Z}^3$, so that $\mathbb{Z}[G] = \mathbb{Z}[u^\pm, v^\pm, w^\pm]$ is the ring of Laurent polynomials on 3 indeterminates. Then $f := 6 - u - u^{-1} - v - v^{-1} - w - w^{-1}$ satisfies the hypotheses of the previous theorem but is not invertible in $\ell^1(G)$. Thus (X_f, G) satisfies the GOE theorem.

GOE Theorems for Principal Algebraic Dynamical Systems (continued)

$$\mathcal{C}_0(G) := \{\xi \colon G \to \mathbb{R} : \lim_{g \to \infty} \xi(g) = 0\} \subset \ell^\infty(G).$$

Definition

An element $f \in \mathbb{Z}[G]$ is weakly expansive if it satisfies:

(C1) $\forall \xi \in \mathcal{C}_0(G), \quad f \star \xi = 0 \implies \xi = 0$;

(C2) $\exists \omega \in \mathcal{C}_0(G)$ such that $f \star \omega = 1_G$.

f invertible in $\ell^1(G) \quad \implies \quad f$ weakly expansive.

Theorem (CCL-2018)

Let G be a countable abelian group and let $f \in \mathbb{Z}[G]$ such that f is weakly expansive and X_f is connected. Then the p.a.d.s. (X_f, G) satisfies the GOE theorem.

Example

Take $g := \mathbb{Z}^3$, so that $\mathbb{Z}[G] = \mathbb{Z}[u^\pm, v^\pm, w^\pm]$ is the ring of Laurent polynomials on 3 indeterminates. Then $f := 6 - u - u^{-1} - v - v^{-1} - w - w^{-1}$ satisfies the hypotheses of the previous theorem but is not invertible in $\ell^1(G)$. Thus (X_f, G) satisfies the GOE theorem. This dynamical system is known as the 3-dimensional harmonic model.

References

[B-2010] L. Bartholdi, *Gardens of Eden and amenability on cellular automata*, J. Eur. Math. Soc. (JEMS) **12** (2010), 241–248.

[BK-2016] L. Bartholdi and D. Kielak, *Amenability of groups is characterized by Myhill's Theorem*, arXiv:1605.09133.

[CC-2010] T. Ceccherini-Silberstein, M. Coornaert, *Cellular automata and groups*, Springer Monographs in Mahtematics, Springer, Berlin, 2010.

[CC-2015] T. Ceccherini-Silberstein, M. Coornaert, *Expansive actions of countable amenable groups, homoclinic pairs, and the Myhill property*, Illinois J. Math. **59** (2015), no. 3, 597–621.

[CC-2016] T. Ceccherini-Silberstein, M. Coornaert, *A Garden of Eden theorem for Anosov diffeomorphisms on tori*, Topology Appl. **212** (2016), 49–56.

[CC-2017] T. Ceccherini-Silberstein, M. Coornaert, *The Garden of Eden theorem: old and new*, arXiv:1707.08898, Handbook of Group Actions (ed. L. Ji, A. Papadopoulos and S. T. Yau), Vol. V, International Press and Higher Education Press, 2018 (to appear).

[CCL-2018] T. Ceccherini-Silberstein, M. Coornaert, H. Li, *Homoclinically expansive actions and a Garden of Eden theorem for harmonic models*, arXiv:1803.03541.

[CMS-1999] T. Ceccherini-Silberstein, A. Machì, and F. Scarabotti, *Amenable groups and cellular automata*, Ann. Inst. Fourier **49** (1999), 673–685.

[Fra-1970] J. Franks, *Anosov diffeomorphisms*, in Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970,

References (continued)

[Gro-1999] M. Gromov, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. (JEMS) **1** (1999), 109–197.

[Li-2017] H. Li, *Garden of Eden and specification*, Ergodic Theory and Dynamical Systems (to appear), arXiv:1708.09012.

[MM-1993] A. Machì and F. Mignosi, *Garden of Eden configurations for cellular automata on Cayley graphs of groups,* SIAM J. Discrete Math. **6** (1993), 44–56.

[Man-1974] A. Manning, *There are no new Anosov diffeomorphisms on tori*, Amer. J. Math. **96** (1974), 422–429.

[Mo-1963] E. F. Moore, *Machine models of self-reproduction*, vol. 14 of Proc. Symp. Appl. Math., American Mathematical Society, Providence, 1963, pp. 17–34.

[My-1963] J. Myhill, *The converse of Moore's Garden-of-Eden theorem*, Proc. Amer. Math. Soc. **14** (1963), 685–686.

[Sch-1995] K. Schmidt, *Dynamical systems of algebraic origin*, vol. 128 of Progress in Mathematics, Birkhäuser Verlag, Basel, 1995.

[Wal-1968] P. Walters, *Topological conjugacy of affine transformations of tori*, Trans. Amer. Math. Soc. **131** (1968), 40–50.