An Introduction to Symbolic Dynamics

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General Notation

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If X is a finite set, $|X|$ denotes the cardinality of X.

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Exercise 1

Show that if $|A| \geq 2$ then $A^{\mathbb{Z}}$ is uncountable.

The Topology on $A^{\mathbb{Z}}$

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Given configurations $x, y \in A^{\mathbb{Z}}$, we put

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$$d(x,y) \in \{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}.$$

We have that $d(x,y) = \dfrac{1}{2^n}$ if and only if

$$(x(k) = y(k) \quad \forall k \in [-n+1, n-1]) \text{ and } (x(n) \neq y(n) \text{ or } x(-n) \neq y(-n)).$$

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For all $x, y, z \in A^{\mathbb{Z}}$, we have that

$$v(x, y) \geq \min(v(x, z), v(z, y)).$$

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We equip $A^{\mathbb{Z}}$ with the topology defined by d. This topology is called the prodiscrete topology on $A^{\mathbb{Z}}$.

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Let $x \in A^{\mathbb{Z}}$. Show that the sets

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with n running over $\mathbb{N}$, form a base of neighborhoods of x. Show that each $V_n(x)$ is both closed and open in $A^{\mathbb{Z}}$.

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Show that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $A^{\mathbb{Z}}$ converges to $y \in A^{\mathbb{Z}}$ if and only if

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Exercise 5

Show that $A^{\mathbb{Z}}$ is compact.

The Topology on $A^{\mathbb{Z}}$ (continued)

Exercise 6

For $k \in \mathbb{Z}$, let $\pi_k \colon A^{\mathbb{Z}} \to A$ be the map defined by $\pi_k(x) = x(k)$ for all $x \in A^{\mathbb{Z}}$. Equip A with its discrete topology. Show that the topology on $A^{\mathbb{Z}}$ is the coarsest topology on $A^{\mathbb{Z}}$ such that all maps π_k are continuous.

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Exercise 7

Show that $A^{\mathbb{Z}}$ is totally disconnected, that is, every non-empty connected subset of $A^{\mathbb{Z}}$ is reduced to a single configuration.

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The shift on $A^{\mathbb{Z}}$ is the map

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Here $A = \{0, 1\}$.

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Proof.

Observe that

$$v(x,y) - 1 \leq v(\sigma(x), \sigma(y)) \leq v(x,y) + 1$$

so that

$$\frac{1}{2} d(x,y) \leq d(\sigma(x), \sigma(y)) \leq 2d(x,y)$$

for all $x, y \in A^{\mathbb{Z}}$. □

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Exercise 10

Show that there exists $x \in A^{\mathbb{Z}}$ *whose positive* σ*-orbit* $\{\sigma^n(x) | n \geq 1\}$ *is dense in* $A^{\mathbb{Z}}$.

Cellular Automata

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- if M_1 and M_2 are memory sets for τ, then $M_1 \cap M_2$ is also a memory set for τ;
- there is a unique memory set for τ with minimal cardinality. This memory set is called the minimal memory set of τ. It is contained in every memory set of τ.

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τ is a cellular automaton with minimal memory set $M = \{0, 1\}$ and local defining map $\mu\colon A^M = A^2 \to A$ given by

$$00 \mapsto 0, \quad 01 \mapsto 0, \quad 10 \mapsto 0, \quad 11 \mapsto 1.$$

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$$(\tau(x))(k) := x(k) + x(k+1) \quad \text{mod } 2 \quad \forall x \in A^{\mathbb{Z}}, k \in \mathbb{Z}.$$

For example, if

$$x = \ldots 11100101001101110000100011 \ldots \text{ then}$$
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τ is a cellular automaton with minimal memory set $M = \{0,1\}$ and local defining map $\mu\colon A^M = A^2 \to A$ given by

$$00 \mapsto 0, \quad 01 \mapsto 1, \quad 10 \mapsto 1, \quad 11 \mapsto 0.$$

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Example 3 (continued)

$x :=$

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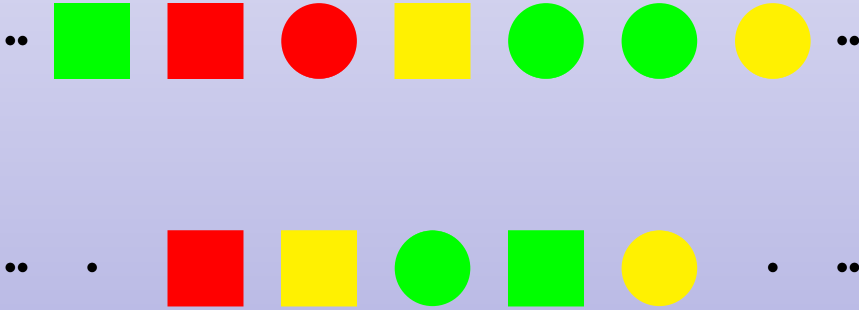

Topological Characterization of Cellular Automata

Theorem 1 (Curtis-Hedlund-Lyndon Theorem)

Let $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a map. Then the following conditions are equivalent:

(a) *τ is a cellular automaton;*

(b) *τ is continuous and $\sigma \circ \tau = \tau \circ \sigma$.*

Topological Characterization of Cellular Automata (continued)

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A subset $X \subset A^{\mathbb{Z}}$ is called a subshift if X is closed in $A^{\mathbb{Z}}$ and $\sigma(X) = X$.

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Corollary 1

If $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a cellular automaton, then $\tau(A^{\mathbb{Z}})$ is a subshift.

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A subset $X \subset A^{\mathbb{Z}}$ is called a subshift if X is closed in $A^{\mathbb{Z}}$ and $\sigma(X) = X$.

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Corollary 2

Let $\tau_1, \tau_2 \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be cellular automata. Then $\tau_1 \circ \tau_2$ is a cellular automaton.

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Corollary 3

Let $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a bijective cellular automaton. Then $\tau^{-1} \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is also a cellular automaton.

Topological Characterization of Cellular Automata (continued)

Exercise 11

Show that the image of the cellular automaton in Example 1 is the subshift $X \subset \{0,1\}^{\mathbb{Z}}$ *consisting of all configurations* $x \in \{0,1\}^{\mathbb{Z}}$ *such that* $(x(k), x(k+1), x(k+2)) \neq (1,0,1)$ *for every* $k \in \mathbb{Z}$.

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Exercise 12

Let $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *be a cellular automaton and let* $n \geq 1$ *be an integer. Show that* $\tau(\mathrm{Fix}(\sigma^n)) \subset \mathrm{Fix}(\sigma^n)$.

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Exercise 13

Let $\tau_1, \tau_2 \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *be cellular automata. Show that if* M_1 *is a memory set for* τ_1 *and* M_2 *is a memory set for* τ_2*, then* $M_1 + M_2$ *is a memory set for* $\tau_1 \circ \tau_2$.

Pre-injectivity

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Definition

One says that a cellular automaton $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is pre-injective if the restriction of τ to each equivalence class of $\sim$ is injective.

The Garden of Eden Theorem

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

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Exercise 14

Show that the cellular automaton in Example 1 is not pre-injective (and hence not injective). Show directly that it is not surjective.

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Show that the cellular automaton in Example 1 is not pre-injective (and hence not injective). Show directly that it is not surjective.

Exercise 15

Show that the cellular automaton in Example 2 is pre-injective but not injective. Show directly that it is surjective.

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Exercise 15

Show that the cellular automaton in Example 2 is pre-injective but not injective. Show directly that it is surjective.

Exercise 16

Show that the cellular automaton in Example 3 is injective (and hence pre-injective). Show directly that it is surjective.

The Garden of Eden Theorem (continued)

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As injectivity implies pre-injectivity, an immediate consequence of the GOE theorem is the following.

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Corollary 4

Every injective cellular automaton $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is surjective (and hence bijective).*

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Corollary 4

Every injective cellular automaton $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is surjective (and hence bijective).

Exercise 17 (An alternative proof)

Let $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an injective cellular automaton. Use Exercise 8 and Exercise 12 to prove that $\tau(\mathrm{Fix}(\sigma^n)) = \mathrm{Fix}(\sigma^n)$ for every integer $n \geq 1$. Use Exercise 9 to conclude that τ is surjective.

Proof of the Garden of Eden Theorem

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Lemma 1

$$|X_{n+m}| \leq |X_n| \cdot |X_m|$$

for all integers $n, m \geq 1$.

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Definition

The entropy of X is

$$\text{ent}(X) := \lim_{n\to\infty} \frac{\log |X_n|}{n}.$$

Proof of the Garden of Eden Theorem (continued)

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Remark

The limit in the definition of $\text{ent}(X)$ is a true limit since the sequence $(\log |X_n|)_{n \geq 1}$ is subadditive by Lemma 1.

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We always have $\text{ent}(X) \leq \log |A|$ since $|X_n| \leq |A|^n$ and hence $\log |X_n| \leq n \log |A|$.

Proof of the Garden of Eden Theorem (continued)

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Let $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton and let $X := \tau(A^{\mathbb{Z}})$ denote the image of τ. Then the following conditions are equivalent:

(a) τ *is surjective;*

(b) $\mathrm{ent}(X) = \log |A|$*;*

(c) τ *is pre-injective.*

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Proof. We can assume $|A| \geq 2$.

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Proof. We can assume $|A| \geq 2$.
(a) $\implies$ (b). Suppose that τ is surjective, that is, $X = A^{\mathbb{Z}}$. Then $X_n = A^n$, so that

$$\text{ent}(X) = \lim_{n\to\infty} \frac{\log|X_n|}{n} = \lim_{n\to\infty} \frac{\log|A^n|}{n} = \lim_{n\to\infty} \frac{n\log|A|}{n} = \log|A|.$$

Proof of the Garden of Eden Theorem (continued)

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(b) $\implies$ (a).
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As X is a closed subset of $A^{\mathbb{Z}}$ (cf. Corollary 1), the set $A^{\mathbb{Z}} \setminus X$ is a non-empty open subset of $A^{\mathbb{Z}}$.

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As X is a closed subset of $A^{\mathbb{Z}}$ (cf. Corollary 1), the set $A^{\mathbb{Z}} \setminus X$ is a non-empty open subset of $A^{\mathbb{Z}}$. Consequently, there exists an integer $n \geq 1$ such that X_n is a proper subset of A^n, so that $|X_n| \leq |A|^n - 1$.

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For every integer $i \geq 1$, we then have

$$|X_{in}| \leq |X_n|^i \leq (|A|^n - 1)^i = |A|^{in} \left(1 - \frac{1}{|A|^n}\right)^i.$$

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This implies

$$\mathrm{ent}(X) = \lim_{i \to \infty} \frac{\log |X_{in}|}{in} \leq \log |A| + \frac{1}{n} \log \left(1 - \frac{1}{|A|^n}\right) < \log |A|.$$

Proof of the Garden of Eden Theorem (continued)

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$$Y := \{x \in A^{\mathbb{Z}} | \, x(k) = a_0 \quad \forall k \in \mathbb{Z} \setminus [1, n]\}.$$

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By the pigeon-hole principle, there exist distinct configurations $y_1, y_2 \in Y$ such that $\tau(y_1) = \tau(y_2)$.

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this implies $|\tau(Y)| < |Y|$.
By the pigeon-hole principle, there exist distinct configurations $y_1, y_2 \in Y$ such that $\tau(y_1) = \tau(y_2)$.
As all configurations in Y are almost equal (they coincide outside of $[1, n]$), this shows that τ is not pre-injective.

Proof of the Garden of Eden Theorem (continued)

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(b) $\Longrightarrow$ (c).

(b) $\implies$ (c).
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Proof of the Garden of Eden Theorem (continued)

(b) $\implies$ (c).
Suppose that τ is not pre-injective.
Then there exist an integer $n \geq 1$ and distinct elements $p, q \in A^{[1,n]}$ that are mutually erasable, i.e., if $x, y \in A^{\mathbb{Z}}$ coincide on $\mathbb{Z} \setminus [1, n]$ and satisfy $x|_{[1,n]} = p$ and $y|_{[1,n]} = q$, then $\tau(x) = \tau(y)$.

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We then deduce that, for every $i \geq 1$,

$$|X_{in}| \leq (|A|^n - 1)^i \, |A|^{2m},$$

where m is such that $[-m, m]$ is a memory set for τ.

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We then deduce that, for every $i \geq 1$,

$$|X_{in}| \leq (|A|^n - 1)^i \, |A|^{2m},$$

where m is such that $[-m, m]$ is a memory set for τ.
This implies

$$\mathrm{ent}(X) = \lim_{i \to \infty} \frac{\log |X_{in}|}{in} \leq \log |A| + \frac{1}{n} \log \left(1 - \frac{1}{|A|^n}\right) < \log |A|.$$

Exercise 18

Compute the entropy of the image subshift of the cellular automaton in Example 1.

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