Some extensions of the Moore-Myhill Garden of Eden Theorem

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Shifts and subshifts

Take:

- a group G,
- a set A (called the alphabet).

The set

$$A^G = \{x \colon G \to A\}$$

is endowed with its prodiscrete topology and the left action of G given by

$$\begin{aligned} G \times A^G &\to A^G \\ (g, x) &\mapsto gx \end{aligned}$$

where

$$gx(h) = x(g^{-1}h) \quad \forall h \in G.$$

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This action is called the *G-shift*. It is continuous w. r. to the prodiscrete topology on A^G.
The space A^G is called the space of configurations or the full shift over the group G and the alphabet A.
A closed G-invariant subset $X \subset A^G$ is called a subshift.

Cellular automata

Definition

Let $X \subset A^G$ be a subshift. A cellular automaton over X is a map

$$\tau \colon X \to X$$

satisfying the following condition:
there exist a finite subset $M \subset G$ and a map $\mu \colon A^M \to A$ such that:

$$(\tau(x))(g) = \mu((g^{-1}x)|_M) \quad \forall x \in X, \forall g \in G,$$

where $(g^{-1}x)|_M$ denotes the restriction of the configuration $g^{-1}x$ to M.

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Such a set M is called a memory set and μ is called a local defining map for τ.

Example: Conway's Game of Life

Here $G = \mathbb{Z}^2$ and $A = \{0, 1\}$.
Life is the cellular automaton

$$\tau \colon \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$$

over the full shift $X = \{0,1\}^{\mathbb{Z}^2}$ obtained by taking $M = \{-1, 0, 1\}^2 \subset \mathbb{Z}^2$ and $\mu \colon A^M \to A$ given by

$$\mu(y) = \begin{cases} 1 & \text{if} \begin{cases} \displaystyle\sum_{m \in M} y(m) = 3 \\ \text{or} \displaystyle\sum_{m \in M} y(m) = 4 \text{ and } y((0,0)) = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

$\forall y \in A^M$.

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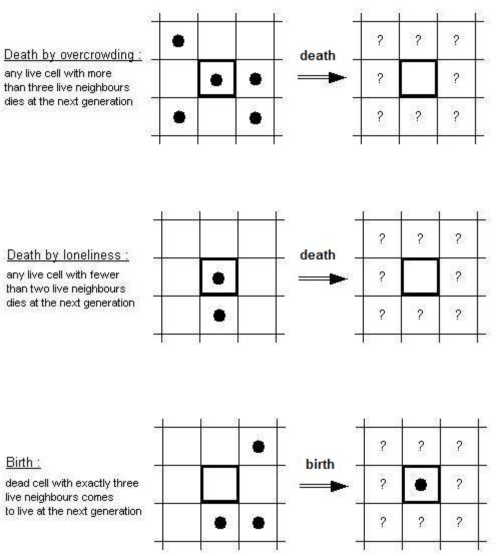

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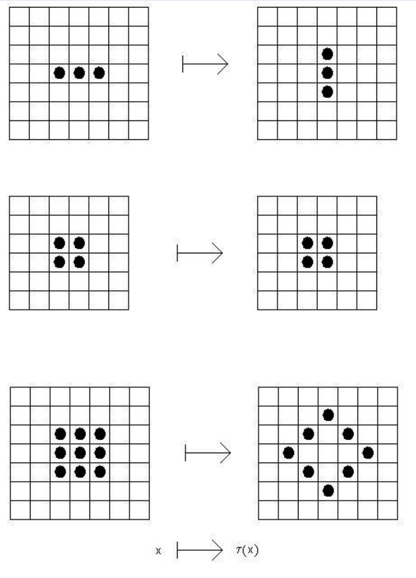

Pre-injectivity

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A cellular automaton $\tau\colon X \to X$ over a subshift $X \subset A^G$ is called pre-injective if:

$$\left.\begin{array}{c} \tau(x) = \tau(x') \\ \text{and} \\ \{g \in G \mid \ x(g) \neq x'(g)\} \text{ is finite} \end{array}\right\} \Longrightarrow x = x'.$$

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The cellular automaton $\tau\colon \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$ associated with Conway's Game of Life is not pre-injective.

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The cellular automaton $\tau\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ defined by

$$\forall x \in \{0,1\}^{\mathbb{Z}}, \forall n \in \mathbb{Z}, \quad \tau(x)(n) = x(n+1) + x(n) \mod 2,$$

is pre-injective. However, it is not injective since the two constant configurations have the same image.

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Let $G = \mathbb{Z}^d$ and let A be a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton defined over the full shift A^G. Then

$$\tau \text{ surjective } \iff \tau \text{ pre-injective.}$$

Moore proved the implication "surjective $\Longrightarrow$ pre-injective" and Myhill proved the converse.

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Corollary

τ *injective* $\Longrightarrow \tau$ *surjective.*

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Sketch of proof of the Moore-Myhill GOE theorem

Consider the cube

$$C_n := \{0, 1, \dots, n-1\}^d \subset \mathbb{Z}^d$$

and the restriction map

$$\pi_n \colon A^{\mathbb{Z}^d} \to A^{C_n}.$$

The entropy of a subset $Y \subset A^{\mathbb{Z}^d}$ is defined by

$$\mathrm{ent}(Y) := \limsup_{n\to\infty} \frac{\log |\pi_n(Y)|}{|C_n|} = \limsup_{n\to\infty} \frac{\log |\pi_n(Y)|}{n^d},$$

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One shows that

$$\tau \text{ surjective } \iff \mathrm{ent}(\tau(A^G)) = \log |A| \iff \tau \text{ pre-injective.}$$

Amenable groups

Definition

The group G is called amenable if there exists a finitely-additive left-invariant probability measure defined on the set $\mathcal{P}(G)$ of all subsets of G, that is, a map $m\colon \mathcal{P}(G) \to [0,1]$ such that

(Amen-1) $m(G) = 1$

(Amen-2) $A \cap B = \varnothing \Rightarrow m(A \cup B) = m(A) + m(B)$

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for all $g \in G$ and $A, B \in \mathcal{P}(G)$.

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- Every abelian group (and, more generally, every solvable group) is amenable.
- Every finitely generated group with subexponential growth is amenable.
- An example of a non-amenable group is provided by the free group on 2 generators.

More generally, every group containing a non-abelian free subgroup is non-amenable.

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Let G be an amenable group and let A be a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton defined over the full shift A^G. Then

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The GOE theorem over amenable groups

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It extends the Moore-Myhill GOE theorem since $\mathbb{Z}^d$ is commutative and hence amenable.

Sketch of proof of the GOE theorem for amenable groups

One uses Følner criterion for amenability: a group G is amenable if and only if it admits a Følner net, i.e., a net $(F_i)_{i \in I}$ of nonempty finite subsets of G such that

$$\lim_i \frac{|F_i \setminus F_i g|}{|F_i|} = 0 \quad \text{for all } g \in G.$$

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Strongly irreducible subshifts of finite type

Let G be a group and let A be a set.

Definition

A subshift $X \subset A^G$ is said to be of finite type if there exist a finite subset $D \subset G$ and a subset $L \subset A^D$ such that

$$X = X(D, L) \overset{def}{=} \{x \in A^G : (g^{-1}x)|_D \in L \text{ for all } g \in G\}.$$

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Definition

A subshift $X \subset A^G$ is said to be strongly irreducible if there exists a finite subset $\Delta \subset G$ with the following property:
if Ω_1 and Ω_2 are finite subsets of G such that there is no element $g \in \Omega_2$ such that the set $g\Delta$ meets Ω_1 then, given any two configurations $x_1, x_2 \in X$, there exists a configuration $x \in X$ such that $x|_{\Omega_1} = x_1|_{\Omega_1}$ and $x|_{\Omega_2} = x_2|_{\Omega_2}$.

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Theorem (F-2003)

Let G be an amenable group and let A be a finite set. Let $\tau \colon X \to X$ be a cellular automaton defined over a strongly irreducible subshift of finite type $X \subset A^G$. Then

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Proof.

Here one shows

$$\tau \text{ surjective} \iff \mathrm{ent}(\tau(X)) = \mathrm{ent}(X) \iff \tau \text{ pre-injective.}$$

□

Example

Let $X = \{x_0, x_1\} \subset \{0,1\}^{\mathbb{Z}}$ be the subshift consisting of the two constant configurations:

$$x_0 = \dots 000000000000 \dots \text{ and } x_1 = \dots 111111111111 \dots$$

Then X is of finite type. The cellular automaton $\tau \colon X \to X$ defined by $\tau(x_0) = \tau(x_1) = x_0$ is pre-injective but not surjective.

Some counterexamples

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Example

Consider the subshift of finite type $X = X(D, L) \subset \{0,1,2\}^{\mathbb{Z}}$, where $D = \{1,2\}$ and $L = \{00, 01, 11, 12, 22\}$. Then the cellular automaton $\tau \colon X \to X$ defined by the substitution rule $12 \mapsto 11$ is injective but not surjective.

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Let $X \subset \{0,1,2\}^{\mathbb{Z}}$ be the subshift consisting of the sequences where the words 01 and 02 are forbidden. Then X is of finite type. The cellular automaton $\tau \colon X \to X$ defined by the substitution rule $\star 0 \mapsto 00$ is surjective but not pre-injective.

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Example

Let $X \subset \{0,1\}^{\mathbb{Z}}$ be the even subshift, i.e., the subshift formed by all sequences in which every chain of 0s which is bounded by two 1 has even length. The subshift X is not of finite type but it is strongly irreducible. Fiorenzi [F-2003] constructed a cellular automaton $\tau \colon X \to X$ which is surjective but not pre-injective.

The Myhill property for strongly irreducible subshifts

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Theorem (CC-2010a)

Let G be an amenable group and let A be a finite set. Let $\tau \colon X \to X$ be a cellular automaton defined over a strongly irreducible subshift $X \subset A^G$. Then

$$\tau \text{ pre-injective} \Longrightarrow \tau \text{ surjective}$$

(Myhill implication).

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(Myhill implication).

Proof.

Here one shows

$$\tau \text{ pre-injective } \implies \mathrm{ent}(\tau(X)) = \mathrm{ent}(X) \implies \tau \text{ surjective.}$$

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Theorem (CC-2010b)

Let G be an amenable group, K a field, and V a finite-dimensional vector space over K. Let $\tau \colon X \to X$ be a linear cellular automaton defined over a strongly irreducible linear subshift of finite type $X \subset V^G$. Then

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The case of the full shift $X = V^G$ had been previously obtained in [CC-2006].

Sketch of proof

Given a Følner net $(F_i)_{i \in I}$ for G, we define the mean dimension $\mathrm{mdim}(Y)$ of a vector subspace $Y \subset V^G$ by

$$\mathrm{mdim}(Y) = \limsup_i \frac{\dim(\pi_{F_i}(Y))}{|F_i|},$$

where $\pi_{F_i} \colon V^G \to V^{F_i}$ is the natural projection map and $\dim(\cdot)$ denotes dimension for finite-dimensional K-vector spaces.

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where $\pi_{F_i} \colon V^G \to V^{F_i}$ is the natural projection map and $\dim(\cdot)$ denotes dimension for finite-dimensional K-vector spaces.
Here one shows

$$\tau \text{ surjective} \iff \mathrm{mdim}(\tau(X)) = \mathrm{mdim}(X) \iff \tau \text{ pre-injective.}$$

In this proof mean dimension plays the role played by entropy in the classical (finite alphabet) case.

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