A Garden of Eden Theorem for Linear Subshifts

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Shifts and subshifts

Take:
- a group G,
- a set A (called the alphabet).

Consider the set

$$A^G = \{x \colon G \to A\}$$

endowed with its prodiscrete topology and the left action of G given by

$$\begin{aligned} G \times A^G &\to A^G \\ (g, x) &\mapsto gx \end{aligned}$$

where

$$gx(h) = x(g^{-1}h) \quad \forall h \in G.$$

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The space A^G is called the space of configurations or the full shift over the group G and the alphabet A.
A closed G-invariant subset $X \subset A^G$ is called a subshift.

Cellular automata

Definition

Let $X \subset A^G$ be a subshift. A cellular automaton over X is a map

$$\tau \colon X \to X$$

satisfying the following condition:
there exist a finite subset $M \subset G$ and a map $\mu \colon A^M \to A$ such that:

$$(\tau(x))(g) = \mu((g^{-1}x)|_M) \quad \forall x \in X, \forall g \in G,$$

where $(g^{-1}x)|_M$ denotes the restriction of the configuration $g^{-1}x$ to M.

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Such a set M is called a memory set and μ is called a local defining map for τ.

Example: Conway's Game of Life

Here $G = \mathbb{Z}^2$ and $A = \{0,1\}$.
Life is the cellular automaton

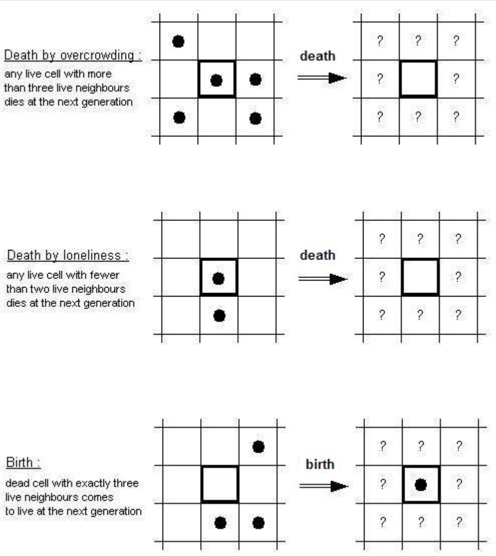
$$\tau\colon \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$$

over the full shift $X = \{0,1\}^{\mathbb{Z}^2}$ obtained by taking $M = \{-1,0,1\}^2 \subset \mathbb{Z}^2$ and $\mu\colon A^M \to A$ given by

$$\mu(y) = \begin{cases} 1 & \text{if} \begin{cases} \displaystyle\sum_{m\in M} y(m) = 3 \\ \text{or} \displaystyle\sum_{m\in M} y(m) = 4 \text{ and } y((0,0)) = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

$\forall y \in A^M$.

Example: Conway's Game of Life



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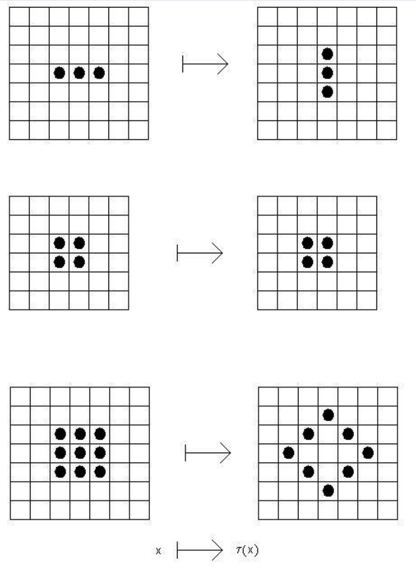

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The group G is called amenable if there exists a finitely-additive left-invariant probability measure defined on the set $\mathcal{P}(G)$ of all subsets of G, that is, a map $m\colon \mathcal{P}(G) \to [0,1]$ such that

(Amen-1) $m(G) = 1$

(Amen-2) $A \cap B = \varnothing \Rightarrow m(A \cup B) = m(A) + m(B)$

(Amen-3) $m(gA) = m(A)$

for all $g \in G$ and $A, B \in \mathcal{P}(G)$.

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- Every abelian group (and, more generally, every solvable group) is amenable.
- Every finitely generated group with subexponential growth is amenable.
- An example of a non-amenable group is provided by the free group on 2 generators.

More generally, every group containing a non-abelian free subgroup is non-amenable.

Definition

A cellular automaton $\tau\colon X \to X$ over a subshift $X \subset A^G$ is called pre-injective if:

$$\left.\begin{array}{c} \tau(x) = \tau(x') \\ \text{and} \\ \{g \in G \mid \ x(g) \neq x'(g)\} \text{ is finite} \end{array}\right\} \Longrightarrow x = x'.$$

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The following theorem is the Garden of Eden Theorem due to Ceccherini-Silberstein, Machì and Scarabotti. It was first proved for $G = \mathbb{Z}^2$ by Moore and Myhill and then extended to finitely generated groups of subexponential growth by Machì and Mignosi.

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Theorem (CMS,1999)

Let G be an amenable group and let A be a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton defined over the full shift A^G. Then

$$\tau \text{ surjective } \iff \tau \text{ pre-injective.}$$

A Garden of Eden theorem for subshifts

Let G be a group and let A be a set.

Definition

A subshift $X \subset A^G$ is said to be **of finite type** if there exist a finite subset $D \subset G$ and a subset $L \subset A^D$ such that

$$X = X(D, L) \stackrel{def}{=} \{x \in A^G : (g^{-1}x)|_D \in L \text{ for all } g \in G\}.$$

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Definition

A subshift $X \subset A^G$ is said to be strongly irreducible if there exists a finite subset $\Delta \subset G$ with the following property:
if Ω_1 and Ω_2 are finite subsets of G such that there is no element $g \in \Omega_2$ such that the set $g\Delta$ meets Ω_1 then, given any two configurations $x_1, x_2 \in X$, there exists a configuration $x \in X$ such that $x|_{\Omega_1} = x_1|_{\Omega_1}$ and $x|_{\Omega_2} = x_2|_{\Omega_2}$.

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Fiorenzi proved the following extension of the Garden of Eden theorem:

Theorem (F, 1999)

Let G be an amenable group and let A be a finite set. Let $\tau : X \to X$ be a cellular automaton defined over a strongly irreducible subshift of finite type $X \subset A^G$. Then

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

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A linear cellular automaton over a linear subshift $X \subset V^G$ is a cellular automaton $\tau \colon X \to X$ which is $\mathbb{K}$-linear.
This is joint work with Tullio Ceccherini-Silberstein:

Theorem (CC)

Let G be an amenable group, $\mathbb{K}$ a field, and V a finite-dimensional vector space over $\mathbb{K}$. Let $\tau \colon X \to X$ be a linear cellular automaton defined over a strongly irreducible linear subshift of finite type $X \subset V^G$. Then

$$\tau \text{ surjective} \iff \tau \text{ pre-injective.}$$

Corollary

τ injective $\Rightarrow$ τ surjective.

Sketch of proof

We use Følner criterion for amenability: a group G is amenable if and only if it admits a Følner net, i.e., a net $(F_i)_{i \in I}$ of nonempty finite subsets of G such that

$$\lim_i \frac{|F_i \setminus F_i g|}{|F_i|} = 0 \quad \text{for all } g \in G.$$

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We define the mean dimension $\operatorname{mdim}(Y)$ of a vector subspace $Y \subset V^G$ by

$$\operatorname{mdim}(Y) = \limsup_i \frac{\dim(\pi_{F_i}(Y))}{|F_i|},$$

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In this proof mean dimension plays the role played by entropy in the classical (finite alphabet) case.

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