A Garden of Eden theorem for Anosov diffeomorphisms on tori

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This is joint work with Tullio Ceccherini-Silberstein.

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Our motivation came from the following phrase of Gromov [Gro-1999, p. 195]:

> *"...the Garden of Eden theorem can be generalized to a suitable class of hyperbolic actions ..."*

Dynamical systems

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The space X is called the phase space of the dynamical system.

Examples of Dynamical systems

Example (Arnold's cat)

This is the d.s. $(\mathbb{T}^2, f)$, where f is the homeomorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ given by

$$\forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{T}^2, \quad f(x) = \begin{pmatrix} x_2 \\ x_1 + x_2 \end{pmatrix}.$$

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We take a finite set A, called the alphabet or the set of states. The associated shift is the d.s. $(A^{\mathbb{Z}}, \sigma)$, where

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is equipped with the topology of pointwise convergence and $\sigma \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by

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An element of $A^{\mathbb{Z}}$ is called a configuration. A subsystem of the shift (i.e., a pair (X, σ), were $X \subset A^{\mathbb{Z}}$ is a closed σ-invariant subspace) is called a subshift.

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Definition

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Of course

$$\boxed{\tau \text{ injective} \Longrightarrow \tau \text{ pre-injective}}$$

but the converse implication is false in general.

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The kernel of τ consists of four points:

$$\mathrm{Ker}(\tau) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\}.$$

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Example (Full shift on $\{0, 1\}$)

The endomorphism τ of the full shift $(A^{\mathbb{Z}}, \sigma)$ on the alphabet

$$A := \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$$

defined by

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- Myhill [Myh-1963] proved $\Longleftarrow$.

The proof consists in showing that

$$\tau \text{ surjective} \iff h_{top}(\tau(A^{\mathbb{Z}}), f) = h_{top}(A^{\mathbb{Z}}, f) \iff \tau \text{ pre-injective},$$

where $h_{top}(X, f)$ denotes the topological entropy of the d.s. (X, f).

The Moore and the Myhill properties

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Definition

A d.s. has the Moore-Myhill property if it has both the Moore and the Myill property.

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Let X be a compact metrizable space and take $f = \mathrm{Id}$. The endomorphisms of (X, f) consist of all continuous maps $\tau \colon X \to X$. Every f-homoclinicity class is reduced to a single point so that each endomorphism of (X, f) is pre-injective. Thus (X, f) has the Moore property. However, (X, f) does not have the Myhill property as soon as X has more than one point.

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The even subshift is the subshift $X \subset \{0, 1\}^{\mathbb{Z}}$ consisting of all bi-infinite sequences $x \colon \mathbb{Z} \to \{0, 1\}$ such that the number of 1s between any two 0s is even.

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Remark

The Moore property is a finiteness condition (i.e., every d.s. (X, f) with X finite has the Moore property) whereas the Myhill property is not (consider a d.s. (X, Id), where X is a discrete finite space with more than one point).

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One says that f is Anosov if the tangent bundle TM of M continuously splits as a direct sum $TM = E_s \oplus E_u$ of two df-invariant subbundles E_s and E_u such that, with respect to some (or equivalently any) Riemannian metric on M, the differential df is exponentially contracting on E_s and exponentially expanding on E_u, i. e., there are constants $C > 0$ and $0 < \lambda < 1$ such that

- $\|df^n(v)\| \leq C\lambda^n \|v\|$,
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for all $x \in M$, $v \in E_s(x)$, $w \in E_u(x)$, and $n \geq 0$.

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Example

Arnold's cat is Anosov. If we identify the tangent space at $x \in \mathbb{T}^2$ with $\mathbb{R}^2$, the two eigenlines of the cat matrix yield $E_u(x)$ and $E_s(x)$.

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Consider a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ with no eigenvalue of modulus 1. Then the map

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is an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$.

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One says that f is the hyperbolic toral automorphism associated with A.

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Theorem (CC-2015a)

Let f be an Anosov diffeomorphism of the n-dimensional torus $\mathbb{T}^n$. Then the d.s. $(\mathbb{T}^n, f)$ has the Moore-Myhill property.

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Result 1 (Franks [Fra-1970], Manning [Man-1974]) Every Anosov diffeomorphisms of $\mathbb{T}^n$ is topologically conjugate to a hyperbolic toral automorphism.

Result 2 (Walters [Wal-1968]) Every endomorphism of a hyperbolic toral automorphism on $\mathbb{T}^n$ is affine, i. e., of the form $x \mapsto Bx + c$, where B is an integral $n \times n$ matrix and $c \in \mathbb{T}^n$.

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A d.s. (X, f) is topologically mixing if, given any two non-empty open subsets $U, V \subset X$, one has $f^n(U) \cap V \neq \varnothing$ for all but finitely many $n \in \mathbb{Z}$.

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A dynamical system (X, f) is expansive if there exists a constant $\delta > 0$ such that, for every pair of distinct points $x, y \in X$, there exists $n = n(x, y) \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq \delta$.
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A subshift $\Sigma \subset A^{\mathbb{Z}}$ is said to be of finite type if there is an integer $m \geq 1$ and a subset $P \subset A^m$ such that X consists of the configurations $x \in A^{\mathbb{Z}}$ that satisfy

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The Myhill property for topologically mixing Anosov diffeomorphisms

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Theorem (CC-2015b)

Suppose that the d.s. (X, f) is expansive and that there exist a finite set A, a topologically mixing subshift of finite type $\Sigma \subset A^{\mathbb{Z}}$, and a uniformly bounded-to-one factor map $\pi \colon \Sigma \to X$. Then the dynamical system (X, f) has the Myhill property.

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Remark

The even subshift satisfies the hypotheses of the previous theorem but does not have the Moore property.

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Let f be a topologically mixing Anosov diffeomorphism of a smooth compact manifold M. Then (M, f) has the Myhill property.

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Remark

All known examples of Anosov diffeomorphisms are topologically mixing.

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