Anick-type resolutions, shuffle algebras, and consecutive pattern avoidance

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which easily yields a formula for generating functions

$$f_{\mathrm{no-}\texttt{SEX}}(t) = \left.\frac{1}{1-(26t+y)}\right|_{y=-t^3} = \frac{1}{1-26t+t^3}.$$

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Here

$$\mathtt{SEXPERT} = \left\{ \begin{matrix} \mathtt{SEX} & \\ & \mathtt{EXPERT} \end{matrix} \right\}$$

is a *cluster*.

Goulden–Jackson cluster method

Theorem (I. P. Goulden & D. M. Jackson '79):
Let P be a set of illegal words in the alphabet X. Then

$$f_{\mathrm{no-P}}(t) = \frac{1}{1 - |X|t + \mathrm{Cl}_{\mathsf{P}}(t, -1)},$$

where $\mathrm{Cl}_{\mathsf{P}}(t, s) = \sum \mathrm{cl}^{\mathsf{P}}_{n,m} t^n s^m$ counts clusters ($\mathrm{cl}^{\mathsf{P}}_{n,m}$ is the number of clusters on n letters formed by m words from P).

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Here we have infinitely many clusters, e.g. EXPERTISE, EXPERTISEXPERTISE, EXPERTISEXPERTISEXPERTISE etc.

Moreover, some words admit many different coverings, e.g. we have the following two clusters

$$\left\{\begin{array}{l}\texttt{EXPERTISE}\\ \qquad\qquad\quad\texttt{EXPERTISE}\end{array}\right\} \text{ and } \left\{\begin{array}{l}\texttt{EXPERTISE}\\ \qquad\qquad\texttt{SEX}\\ \qquad\qquad\quad\texttt{EXPERTISE}\end{array}\right\}.$$

Avoidance of SEX and EXPERTISE

Observation: Contributions of the two clusters

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After cancellations: clusters that contribute are SEX, EXPERTISE, SEXPERTISE, EXPERTISEX, SEXPERTISEX, so that

$$f_{\text{no}-\texttt{SEX},\text{no}-\texttt{EXPERTISE}}(t) = \frac{1}{1 - 26t + t^3 + t^9 - 2t^{10} + t^{11}}.$$

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Example:
`EXPERTISEXPERTISE`, even though can be represented as a link of two illegal words, is not a 2-chain because its proper beginning `EXPERTISEX` is already a 2-chain! It's not a 3-chain either, because the first and the third illegal words are linked.

Anick chains

Theorem (D. J. Anick '86): We have

$$f_{\mathrm{no-P}}(t) = \frac{1}{1 - |X|t + \mathrm{C_P}(t, -1)},$$

where $\mathrm{C_P}(t, s) = \sum \mathrm{c}^{\mathsf{P}}_{n,m} t^n s^m$ counts chains ($\mathrm{c}^{\mathsf{P}}_{n,m}$ is the number of m-chains on n letters).

Anick resolution

Proof: Denote by A the associative algebra with generators X and relations $\mathrm{P} = 0$. Also, denote by C_m the vector space with a basis of m-chains. Then there exists a chain complex

$$\ldots \to C_n \otimes A \to C_{n-1} \otimes A \to \ldots \to C_1 \otimes A \to C_0 \otimes A \to A \to 0,$$

whose homology is concentrated in the rightmost term and is one-dimensional. Boundary maps move "tails" through the tensor product: $\partial(w't \otimes a) = w' \otimes ta$.

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Compute (graded) Euler characteristics of this complex:

$$(1 - C_0(t) + C_1(t) - \ldots)A(t) = 1.$$

Clearly, $1 - C_0(t) + C_1(t) - \ldots = 1 - mt + \mathrm{C}_{\mathrm{P}}(t, -1)$, and $A(t)$ enumerates words that avoid P. $\square$

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For enumeration, exponential generating functions are used, e.g.

$$f_{\mathrm{no}-132}(t) = 1 + \sum_{n \geq 1} \frac{a_{\mathrm{no}-132}(n)}{n!} t^n.$$

Pattern avoidance in permutations

Theorem (I. P. Goulden & D. M. Jackson '79):

$$f_{\mathrm{no}-123}(t) = \frac{1}{1 - t + \frac{t^3}{3!} - \frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \dots}.$$

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Theorem (S. Elizalde & M. Noy '03):

$$f_{\text{no}-132}(t) = \frac{1}{1 - \int_0^t e^{-u^2/2}\, du}.$$

Shuffle product of graded vector spaces

Wanted: a materialization on the level of vector spaces for the product of *exponential* generating functions; on the level of coefficients,

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For two graded $\Bbbk$-vector spaces $A = \oplus_{n\geq 1} A_n$ and $B = \oplus_{n\geq 1} B_n$, their shuffle product $A \boxtimes B$ is defined as the graded vector space $C = \oplus_{n\geq 1} C_n$ with

$$C_n = \bigoplus_{k+l=n} \Bbbk \mathrm{Sh}(k,l) \otimes A_k \otimes B_l,$$

where $\mathrm{Sh}(k,l)$ is the set of all (k,l)-shuffles in S_n. It's what we want for generating functions, since $|\mathrm{Sh}(k,l)| = \binom{k+l}{k}$.

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Generalisation: let P be a set of illegal patterns, and let $A_{n,\mathrm{P}}$ be the linear span in $\Bbbk S_n$ of all P-avoiding permutations. Then A_{P} is a shuffle algebra which is the quotient of the free algebra by the ideal generated by P.

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If we start with the free shuffle algebra with several generators, we shall end up with the notion of *coloured patterns* (Mansour '01); all our further statements remain.

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Example: for $\mathsf{P} = \{123\}$ we get 1, 123, $\left\{\begin{matrix}123 & \\ & 234\end{matrix}\right\}$, $\left\{\begin{matrix}123 & & \\ & 234 & \\ & & 456\end{matrix}\right\}$, ...

Note that 12345 is neither a 2-chain (as 1234 is already a 2-chain) nor a 3-chain (as 123 and 345 are linked).

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Denote by A the shuffle algebra with one generator whose relations are all illegal patterns. Also, denote by C_m the vector space with a basis of m-chains. Then there exists a chain complex

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Conjecture (S. Elizalde '03): For a pattern τ without self-overlaps, the number of permutations avoiding τ depends only on the first and the last element of τ.

Thank you for your patience!