Compatible associative products and trees

Vladimir Dotsenko

Dublin Institute for Advanced Studies and Trinity College Dublin

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Poisson manifolds

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Properties:

- anti-symmetry: $\{f, g\} = -\{g, f\}$,
- linearity: $\{f, ag_1 + bg_2\} = a\{f, g_1\} + b\{f, g_2\}$ for scalars a, b,
- Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$,
- Leibniz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

Two examples

Example 1 (Kostant–Kirillov Poisson bracket): Let $\mathfrak{g}$ be a Lie algebra, then $\mathfrak{g}^*$ is a Poisson manifold: the space of linear functions on $\mathfrak{g}^*$ is $\mathfrak{g}$, so we have a Lie bracket

$$\{g_1, g_2\}(\xi) = \xi([g_1, g_2]),$$

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Example 2 (constant bracket): In the previous example, fix $\gamma \in \Lambda^2\mathfrak{g}^*$, and let

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For instance, if $\mathfrak{g}$ is the 2-dimensional solvable Lie algebra, $\mathfrak{g} = \mathrm{span}\{p, q \mid [p, q] = q\}$, we have the Kostant–Kirillov bracket $\{p, q\} = q$, and a constant bracket $\{p, q\} = 1$.

(Autonomous) Hamiltonian formalism

Let M be a Poisson manifold. To every function $f \in C^\infty(M)$, we relate the corresponding *Hamiltonian vector field* $X_f \in \Gamma(TM)$ satisfying

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$H \in C^\infty(M)$ a function (*energy*, *Hamiltonian of the system*). Hamilton evolution equation associated to H:

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Example: $\mathfrak{g}$ from the previous slide, $\{p, q\} = 1$:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \{H, p\} = -\frac{\partial H}{\partial q},$$
$$\frac{\mathrm{d}q}{\mathrm{d}t} = \{H, q\} = \frac{\partial H}{\partial p}.$$

Integrable systems (for algebraists)

Two functions $f, g \in C^{\infty}(M)$ are said to be *in involution* if $\{f, g\} = 0$. A set of functions is said to be *involutive* if any two elements of this set are in involution.

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A "system" (Poisson manifold) is said to be (Liouville) integrable if it admits a maximal independent involutive set of functions $\{F_1, \ldots, F_{n-r}\}$. [Here $n = \dim M$, and $2r$ is the rank of the Poisson structure at generic point.]

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It can be shown that the coordinates t_i can be obtained by algebraic operations, inverting functions and integration. Thus, *integrable systems are solvable in quadratures*, hence the name.

Bi-Hamiltonian systems — 1

Assume that M has two Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$. These brackets are said to be compatible if $a\{\cdot,\cdot\}_1 + b\{\cdot,\cdot\}_2$ is a Poisson bracket for any choice of scalars a, b.
Such a manifold M is called *bi-Hamitonian*, and every vector field on M which is Hamiltonian with respect to both structures — a *bi-Hamiltonian vector field*.

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Let X be a bi-Hamiltonian vector field, that is $X = \{F_1,\cdot\}_1 = \{F_2,\cdot\}_2$. Then F_1 and F_2 are in involution with respect to both Poisson structures:

$$\{F_1, F_2\}_1 = \{F_2, F_2\}_2 = 0,$$
$$\{F_1, F_2\}_2 = -\{F_2, F_1\}_2 = -\{F_1, F_1\}_1 = 0.$$

Bi-Hamiltonian systems — 2

Lenard recursion formula: Generalizing the previous example, we call a sequence of functions $F_0, F_1, F_2, \ldots$ a *bi-Hamiltonian* hierarchy if

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If we assume in addition that $\{F_0, \cdot\}_2 = 0$, these conditions together mean that we require $F_\lambda = F_0 + F_1\lambda + F_2\lambda^2 + \dots$ to be in the Poisson centre for the *generic bracket* $\lambda\{\cdot, \cdot\}_1 - \{\cdot, \cdot\}_2$.

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(Lenard '67, published by Gardner–Greene–Kruskal–Miura '74, studied in depth by Magri '78 and Gel'fand–Dorfman '79. . .)

Example: KdV

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where $H_0 = \int (u^3 - \frac{1}{2}u_x^2)\, dx$ and $P_0 = \frac{\partial}{\partial x}$ defines the Poisson bracket

$$\{F, G\}_0 = \int \frac{\delta F}{\delta u} P_0 \frac{\delta G}{\delta u}\, dx.$$

($\frac{\delta}{\delta u}$ is the Frèchet derivative, $\frac{\delta \int f\, dx}{\delta u} = \sum_{k \geq 0} (-\frac{\mathrm{d}}{\mathrm{d}x})^k \frac{\partial f}{\partial u^{(k)}}$.)

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$F_3 = \int (\frac{5u^4}{2} - 5uu_x^2 + \frac{u_{xx}^2}{2})\, dx$, ...
Note that $\frac{\delta F_0}{\delta u} = \frac{1}{2}$, so F_0 is in the Poisson centre for $\{\cdot, \cdot\}_0$.

Examples arising from Lie algebras

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Remark 1: if γ is a coboundary, $\gamma(g_1, g_2) = \zeta([g_1, g_2])$ for some $\zeta \in \mathfrak{g}^*$, then $F_\lambda(\xi) = F(\xi + \lambda\zeta)$ is in the generic centre, if F is in the Poisson centre for the Kostant–Kirillov bracket $\rightsquigarrow$ shift of argument method (Mishchenko–Fomenko '79, Vinberg '90 ...). Recently used to study generalized Gaudin hamiltonians (Rybnikov '06), G-opers with irregular singularity (Feigin–Frenkel–Rybnikov '07) etc.

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Remark 2: if $\mathfrak{g}$ is the algebra of vector fields on the circle, and γ is the 2-cocycle defining the Virasoro algebra, one obtains the above bi-Hamiltonian interpretation of KdV. ($P_0 = \frac{\partial}{\partial x}$ produces the Kostant–Kirillov bracket, while $P_1 = \frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2u_x$ is responsible for the Virasoro cocycle.)

Algebras with two compatible Lie brackets

Definition: $\mathfrak{g}$ is an algebra with two compatible Lie brackets, if there are two brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ on $\mathfrak{g}$ such that $a\{\cdot,\cdot\}_1 + b\{\cdot,\cdot\}_2$ is a Lie bracket for any choice of scalars a, b.

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If $\mathfrak{g}$ is an algebra with two compatible Lie brackets, $\mathfrak{g}^*$ has a natural structure of a bi-Hamiltonian manifold: two Kostant–Kirillov brackets are compatible.

Algebras with two compatible products

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Similarly, we call A an algebra with two compatible products, if there are two products $(\cdot \star_1 \cdot)$ and $(\cdot \star_2 \cdot)$ on A such that the product $a(\cdot \star_1 \cdot) + b(\cdot \star_2 \cdot)$ is associative for any choice of scalars a, b.

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Claim: if A is an algebra with two compatible products, then the operations $(a \star_1 b) - (b \star_1 a)$ and $(a \star_2 b) - (b \star_2 a)$ make it an algebra with two compatible Lie brackets.

Examples of compatible products

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Exercise: check the compatibility (and find a conceptual explanation).

Further examples

Example: A_k is the space of polynomials in t of degree at most $k-1$, $\alpha_1(t)$ and $\alpha_2(t)$ two polynomials of degree k without common roots. Then, by Euclid,

$$f(t)g(t) = x_1(t)\alpha_1(t) + x_2(t)\alpha_2(t)$$

for unique pair $(x_1(t), x_2(t)) \in A_k \times A_k$. We let $f(t) \star_i g(t) = x_i(t)$.

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Example: If A is an algebra with two compatible products and B is an associative algebra then $A \otimes B$ is an algebra with two compatible products. Consequently, $\mathrm{Mat}_n(A)$ is an algebra with two compatible brackets.This descends on GL_n-invariants, and leads to bi-Hamiltonian interpretations of some "integrable ODEs on associative algebras" (Olver–Sokolov '98, Mikhailov–Sokolov '99).

Results on algebras with two compatible products

Towards the classification: Odesskii–Sokolov '05, classification of ways to make $\oplus_i \mathrm{Mat}_{n_i}$ an algebra with two compatible products. The arising combinatorial data is a representation of a quiver associated to an affine simply laced Dynkin diagram (with some additional marking of vertices).

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YES!

Main theorem

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The vector space $\Bbbk\,\mathrm{RT}(S)$ has two compatible products; with those products it becomes the free algebra with two compatible products generated by S.

Corollary (on Catalan numbers)

The component of degree n in the free algebra with two compatible brackets generated by S has dimension $\frac{1}{n+1}\binom{2n}{n}|S|^n$.

Compatible products of trees

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$$T_1 \star_2 T_2 = \sum_{g\colon [k] \to \mathrm{Int}(T_2)} T_1 \dashv^g T_2,$$

Compatible products of trees

Let $T_1, T_2 \in \mathrm{RT}(S)$. Assume that the root of T_1 has k children. Define the products $T_1 \star_1 T_2$ and $T_1 \star_2 T_2$ as follows:

$$T_1 \star_1 T_2 = \sum_{f\colon [k] \to \mathrm{Vertices}(T_2)} T_1 \dashv^f T_2,$$

$$T_1 \star_2 T_2 = \sum_{g\colon [k] \to \mathrm{Int}(T_2)} T_1 \dashv^g T_2,$$

where the tree $T_1 \dashv^f T_2$ is obtained as follows. We split T_1 into k parts $T_1[1], \ldots, T_1[k]$, and for a vertex v of T_2 with

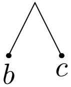
$$f^{-1}(v) = \{i_1 < \ldots < i_s\},$$

we graft $T_1[i_1], \ldots, T_1[i_s]$ at vertex v (keeping the label of v) to the left of all the children of v in T_2.

Example

For the trees

$T_1 =$ 

, $T_2 =$ 

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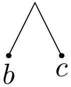
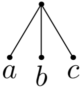
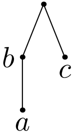
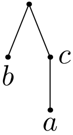
$T_1 =$  , $T_2 =$ 

the product $T_1 \star_1 T_2$ is equal to

 $+$  $+$ 

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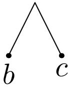
$,\quad T_2 =$ 

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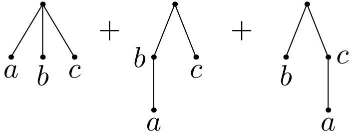
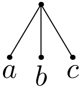


while the product $T_1 \star_2 T_2$ is equal to



Proof strategy for compatibility

Observation 1: The compatibility condition can be rewritten in the form

$$(T_1 \star_2 T_2) \star_1 T_3 - T_1 \star_2 (T_2 \star_1 T_3) = T_1 \star_1 (T_2 \star_2 T_3) - (T_1 \star_1 T_2) \star_2 T_3.$$

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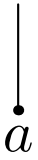
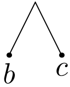
Observation 2: For two products on the space of trees that we defined both the left hand side and the right hand side have positive integer coefficients.

Example

For the trees

$T_1 =$  $, \quad T_2 =$  $, \quad T_3 =$ 

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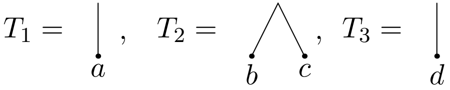

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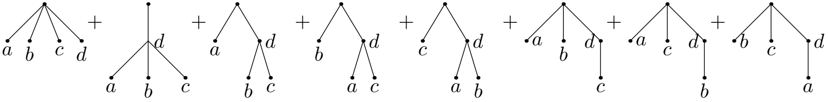

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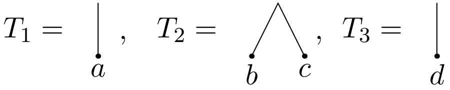

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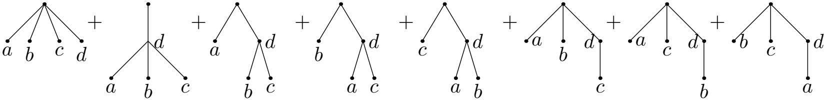

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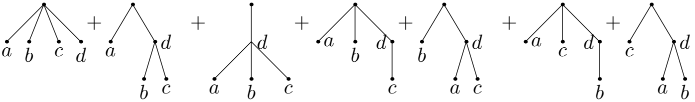

Example

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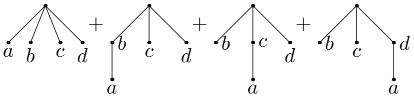

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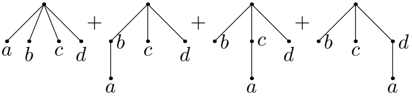
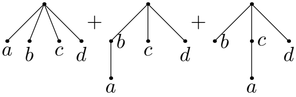


and the product $(T_1 \star_1 T_2) \star_2 T_3$ is



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- free algebra with two compatible products generated by S has the same dimensions of graded components as $\Bbbk\,\mathrm{RT}(S)$. Original proof was using a result of Strohmayer '07 that involved heavy homological machinery (Koszul duality for operads), however now it is clear that using Gröbner bases for operads (D.–Khoroshkin '08) can simplify the proof substantially.

Remarks

The first product $T_1 \star_1 T_2$ on $\mathbb{Q}\,\mathrm{RT}(S)$ was defined by Grossman and Larson '89 in their works about Hopf algebras based on trees (combinatorial Hopf algebras describing solving differential equations etc.). Our results yield an easy proof of their theorem stating that the algebra of planar rooted trees is a free associative algebra generated by trees whose root has only one child.

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Our strategy to construct a compatible product (grafting to internal vertices only) applies to a variety of cases of algebras based on trees, e.g. numerous "Hopf algebras of renormalization" (Brouder–Frabetti '03, Connes–Kreimer '98, Foissy '02).

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(Vague) question: relate renormalization techniques to bi-Hamiltonian integrability.

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Even better news: for free bi-Hamiltonian algebras, the corresponding dimensions are $(n+1)^{n-1}$ (*op. cit.*), which makes one think of diagonal harmonics...

Thank you for your patience!