FREE LIE ALGEBRAS ARE NOT LILY BIALGEBRAS

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The purpose of this short note is to show that the notion of $Lie^c - Lie$-bialgebra in [1] based on the *Lily* compatibility relation between the Lie bracket and the Lie cobracket δ

$$(1)\quad \delta([v,w]) = \frac{1}{2}([v_{(1)},w]\otimes v_{(2)}+v_{(1)}\otimes[v_{(2)},w]+[v,w_{(1)}]\otimes w_{(2)}+w_{(1)}\otimes[v,w_{(2)}])+2(v\otimes w-w\otimes v)$$

(with Sweedler's notation $\delta(v) = v_{(1)} \otimes v_{(2)}$) does not give a good triple of operads. More precisely, the condition `(H1)` of [1] fails: there is no way to introduce such a structure on a free Lie algebra. (The mistake made in [1] is that the relation above is checked only for elements of $Lie(V)$ of the form $[X,z]$, where $\in V$, while in order to rewrite everything in this form, coherence with the Jacobi identity should be checked, and it is not done.)

Let us more generally call a $Lie^c - Lie$ bialgebra a $Lily_{a,b}$-bialgebra, if the following condition is satisfied:

$$(2)\quad \delta([v,w]) = a([v_{(1)},w]\otimes v_{(2)}+v_{(1)}\otimes[v_{(2)},w]+[v,w_{(1)}]\otimes w_{(2)}+w_{(1)}\otimes[v,w_{(2)}])+b(v\otimes w-w\otimes v).$$

(A *Lily*-bialgebra is clearly a $Lily_{\frac{1}{2},2}$-bialgebra.)

Theorem 1. *The free Lie algebra $Lie(V)$ has no structure of a Lie^c coalgebra (with V being the space of primitive elements) for which the relation (2) is satisfied, unless $b = 0$. In particular, the relation Lily does not give rise to a good triple of operads.*

Proof. Assume the contrary. Let us take linearly independent elements $x, y, z, t \in V$. Clearly, $\delta(x) = \delta(y) = \delta(z) = \delta(t) = 0$. Coproducts of all Lie monomials in x, y, z, t can be computed recursively, using the condition (2), as follows. First, we have

$$\delta([x,y]) = b(x\otimes y - y\otimes x), \quad \delta([z,t]) = b(z\otimes t - t\otimes z).$$

This implies

$$\begin{aligned}\delta([y,[z,t]]) &= ab([y,z]\otimes t + z\otimes[y,t] - [y,t]\otimes z - t\otimes[y,z]) + b(y\otimes[z,t] - [z,t]\otimes y),\\ \delta([[z,t],x]) &= ab([z,x]\otimes t + z\otimes[t,x] - [t,x]\otimes z - t\otimes[z,x]) + b([z,t]\otimes x - x\otimes[z,t]).\end{aligned}$$

Finally,

$$\begin{aligned}\delta([[y,[z,t]],x]) = a(&ab([[y,z],x]\otimes t + [z,x]\otimes[y,t] - [[y,t],x]\otimes z - [t,x]\otimes[y,z])+\\ &+ b([y,x]\otimes[z,t] - [[z,t],x]\otimes y)+\\ &+ ab([y,z]\otimes[t,x] + z\otimes[[y,t],x] - [y,t]\otimes[z,x] - t\otimes[[y,z],x])+\\ &+ b(y\otimes[[z,t],x] - [z,t]\otimes[y,x]))+\\ &+ b([y,[z,t]]\otimes x - x\otimes[y,[z,t]]),\end{aligned}$$

$$\begin{aligned}\delta([[[z,t],x],y]) = a(&ab([[z,x],y]\otimes t + [z,y]\otimes[t,x] - [[t,x],y]\otimes z - [t,y]\otimes[z,x])+\\ &+ b([[z,t],y]\otimes x - [x,y]\otimes[z,t])+\\ &+ ab([z,x]\otimes[t,y] + z\otimes[[t,x],y] - [t,x]\otimes[z,y] - t\otimes[[z,x],y])+\\ &+ b([z,t]\otimes[x,y] - x\otimes[[z,t],y]))+\\ &+ b([[z,t],x]\otimes y - y\otimes[[z,t],x]),\end{aligned}$$

and

$$\delta([[x,y],[z,t]]) = a(b([x,[z,t]]\otimes y - [y,[z,t]]\otimes x) + b(x\otimes[y,[z,t]] - y\otimes[x,[z,t]])+ \\ + b([[x,y],z]\otimes t - [[x,y],t]\otimes z) + b(z\otimes[t,[x,y]] - t\otimes[z,[x,y]]))+ \\ + b([x,y]\otimes[z,t] - [z,t]\otimes[x,y]).$$

Therefore,

$$0 = \delta([[x,y],[z,t]] + [[y,[z,t]],x] + [[[z,t],x],y]) = \\ = a^2b([[z,x],y]\otimes t - [[t,x],y]\otimes z + z\otimes[[t,x],y] - t\otimes[[z,x],y]+ \\ + [[y,z],x]\otimes t - [[y,t],x]\otimes z + z\otimes[[y,t],x] - t\otimes[[y,z],x])+ \\ + ab(-2[x,y]\otimes[z,t] + 2[z,t]\otimes[x,y] + 2[x,[z,t]]\otimes y - 2[y,[z,t]]\otimes x + 2x\otimes[y,[z,t]] - 2y\otimes[x,[z,t]]+ \\ + [[x,y],z]\otimes t - [[x,y],t]\otimes z + z\otimes[t,[x,y]] - t\otimes[z,[x,y]])+ \\ + b([y,[z,t]]\otimes x - x\otimes[y,[z,t]] + [[z,t],x]\otimes y - y\otimes[[z,t],x] + [x,y]\otimes[z,t] - [z,t]\otimes[x,y]).$$

Using the Jacobi identity for the bracket, the latter can be rewritten as

$$0 = \delta([[x,y],[z,t] + [[y,[z,t]],x] + [[[z,t],x],y]) = \\ = a^2b(-[[x,y],z]\otimes t + [[x,y],t]\otimes z - z\otimes[[x,y],t] + t\otimes[[x,y],z])+ \\ + ab(-2[x,y]\otimes[z,t] + 2[z,t]\otimes[x,y] + 2[x,[z,t]]\otimes y - 2[y,[z,t]]\otimes x + 2x\otimes[y,[z,t]] - 2y\otimes[x,[z,t]]+ \\ + [[x,y],z]\otimes t - [[x,y],t]\otimes z + z\otimes[t,[x,y]] - t\otimes[z,[x,y]])+ \\ + b([y,[z,t]]\otimes x - x\otimes[y,[z,t]] + [[z,t],x]\otimes y - y\otimes[[z,t],x] + [x,y]\otimes[z,t] - [z,t]\otimes[x,y]) = \\ = (ab - a^2b)(([[x,y],z]\otimes t - [[x,y],t]\otimes z + z\otimes[[x,y],t] - t\otimes[[x,y],z])+ \\ + (b-2ab)([y,[z,t]]\otimes x - x\otimes[y,[z,t]] + [[z,t],x]\otimes y - y\otimes[[z,t],x] + [x,y]\otimes[z,t] - [z,t]\otimes[x,y]).$$

This expression is in the tensor square of $Lie(V)$, so since x, y, z, t are linearly independent generators, this expression is equal to 0 if and only if we have both $ab = a^2b$ and $b = 2ab$, thus $b = 0$ or if $b \neq 0$ (as in the case of *Lily*-bialgebras), $a = a^2$ and $1 = 2a$, and there are no choices for a. □

References

[1] Jean-Louis Loday, *Generalized bialgebras and triples of operads*, Astérisque **320** (2008), x+116 pp.