Hierarchies of identities for differential operators and moduli spaces of curves

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joint with Sergey Shadrin (Amsterdam) and Bruno Vallette (Nice)

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Motivation

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Often presented together with an auxiliary operation

$$[a, b] = \Delta(ab) - \Delta(a)b - a\Delta(b),$$

but we shall not need it.

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One of recent results (Drummond-Cole & Vallette '09) reveals a relationship between the homotopy theory for BV-algebras and moduli spaces of curves with marked points.

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Additional requirement: each Δ_k is a differential operator of order at most k. (Needed if we do not want any further higher structures!)

Differential operators on commutative algebras

Definition (Grothendieck, EGA IV). Let A be an associative commutative algebra. A differential operator of order at most 0 is an operator of the form $f \cdot (-) : g \mapsto f \cdot g$ for some $f \in A$.

Furthermore, a linear operator $D: A \to A$ is a *differential operator of order at most* p if $[D, f \cdot (-)]$ is a differential operator of order at most $p - 1$ for every $f \in A$.

Differential operators on commutative algebras

Definition (Koszul, 1980s). Let A be an associative commutative algebra. The hierarchy of brackets $\langle -,-,\dots,-\rangle_p^D\colon A^{\otimes p}\to A$ associated to a linear operator $D\colon A\to A$ is defined recursively by $\langle f\rangle_1^D := D(f)$ and

$$\langle f_1,\dots,f_{p-1},f_p,f_{p+1}\rangle_{p+1}^D = \langle f_1,\dots,f_{p-1},f_pf_{p+1}\rangle_p^D - \\ - \langle f_1,\dots,f_{p-1},f_p\rangle_p^D f_{p+1} - f_p\langle f_1,\dots,f_{p-1},f_{p+1}\rangle_p^D.$$

An operator D is a *differential operator order at most p* if the bracket $\langle -,-,\dots,-\rangle_{p+1}^D$ is identically equal to zero.

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For algebras with unit, the two definitions are equivalent if we restrict ourselves to operators that annihilate the unit. In this talk, we stick to the second definition.

A hierarchy of formulas

For a differential operator D of order at most p, any $n \geq p+1$, and integers $d_0, \dots, d_n \geq 0$ with $p + d_0 + d_1 + \cdots + d_n = n-1$,

$$\binom{n-2}{d_0+p-1, d_1, \dots, d_n} D(f_1 f_2 \cdots f_n) +$$

$$+ \sum_{i+j=p-2} (-1)^{j+1} \binom{|I|-2}{i, d_{i_1}, \dots, d_{i_r}} \binom{|J|-1}{j, d_0, d_{j_1}, \dots, d_{j_s}} D(f_I) f_J +$$

$$+ (-1)^p \sum_{m=1}^{n} \binom{n-2}{d_0, \dots, d_m+p-1, \dots, d_n} f_1 \cdots D(f_m) \cdots f_n = 0,$$

where in the last sum the summation is also over all $I = \{i_1, \dots, i_r\}$, $J = \{j_1, \dots, j_s\}$ with $r \geq 2$, $I \sqcup J = \{1, \dots, n\}$, and $i + d_{i_1} + \ldots + d_{i_r} = |I| - 2$, $j + d_0 + d_{j_1} + \ldots + d_{j_s} = |J| - 1$.

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- For a differential operator D of order at most 2, the hierarchy includes

$$D(f_1 f_2 \cdots f_n) = \sum_{1 \le i < j \le n} D(f_i f_j) f_1 \cdots \widehat{f_i} \cdots \widehat{f_j} \cdots f_n - \\ - (n-2) \sum_{m=1}^{n} D(f_m) f_1 \cdots \widehat{f_m} \cdots f_n,$$

  but also includes infinitely many other ones.

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  - (also can include projections $\pi\colon \overline{\mathscr{M}}_{g,n+1}\to\overline{\mathscr{M}}_{g,n}$ forgetting the last marked point, but we shall ignore them most of the time).

Modular operads

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An *algebra* over a modular operad $\mathscr{P}$ is a vector space V with a scalar product together with a morphism of modular operads $\mathscr{P} \to \mathrm{End}_V$.

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Cohomological field theories

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In other words, a cohomological field theory on a vector space V is a collection of (co)homology classes

$$\alpha_{g,n} \in \mathrm{Hom}(H_\bullet(\overline{\mathscr{M}}_{g,n}), V^{\otimes n}) \simeq H^\bullet(\overline{\mathscr{M}}_{g,n}) \otimes V^{\otimes n}$$

behaving well with respect to pushforwards via σ and ρ.

Topological field theories

A cohomological field theory is said to be a *topological field theory* if all the classes $\alpha_{g,n}$ are of cohomological degree 0. In this case, they all are determined by $\alpha_{0,3}$, which, viewed as an element of $H^\bullet(\overline{\mathscr{M}}_{0,3}) \otimes V^{\otimes 3} \simeq \mathrm{Hom}(V^{\otimes 2}, V)$, should define a commutative associative product on V.

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If we are only interested in the genus 0 part of a CohFT / TFT, we can "eliminate" the scalar product, using it to identify $V^{\otimes(n+1)}$ with $\mathrm{Hom}(V^{\otimes n}, V)$. In this case $\alpha_{0,n+1}$ becomes the $(n-1)$-fold iterated commutative product on V.

Givental group action on CohFT's

Let V be a vector space with a scalar product η. The space of Laurent series with coefficients in V has a symplectic structure

$$\langle v \otimes f(z), w \otimes g(z) \rangle = \eta(v,w)\,\mathrm{Res}(f(-z)g(z)).$$

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The Givental Lie algebra is the Lie algebra of the "upper triangular subgroup" of the group of symplectomorphisms of that structure. It consists of all series $r_1 z + r_2 z^2 + \dots$, where $r_l \in \mathrm{End}(V)$ is symmetric for odd l and skew-symmetric for even l (with respect to the scalar product η).

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An action of the Givental Lie algebra on CohFT's is defined using "tautological classes" of moduli spaces. Main characters: tautological classes $\psi_1, \ldots, \psi_n \in H^2(\overline{\mathscr{M}}_{g,n})$, the Chern classes of tautological line bundles $\mathbb{L}_1, \ldots, \mathbb{L}_n$ whose fibres are tangent lines at respective marked points.

Givental group action on CohFT's

The Givental Lie algebra action on cohomological field theories takes the system of classes $\alpha_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes V^{\otimes n}$ to the system of classes $(\widehat{r_l z^l}.\alpha)_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes V^{\otimes n}$ given by the formula

$$
\begin{aligned}
(\widehat{r_l z^l}.\alpha)_{g,n} &:= \sum_{m=1}^{n} (\alpha_{g,n} \cdot \psi_m^l) \circ_m r_l + \\
&\quad + \frac{1}{2}\left(\sum_{i=0}^{l-1} (-1)^{i+1} \left(\sigma_*(\alpha_{g-1,n+2} \cdot \psi_{n+1}^i \psi_{n+2}^{l-1-i}), \eta^{-1} r_l \right) + \right. \\
&\quad \left. + \sum_{i+j=l-1} (-1)^{j+1} \left(\rho_*(\alpha_{g_1,|I|+1} \cdot \psi_{|I|+1}^i \otimes \alpha_{g_2,|J|+1} \cdot \psi_{|J|+1}^j), \eta^{-1} r_l \right) \right).
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$$(\widehat{r_l z^l}.\alpha)_{g,n} := \sum_{m=1}^{n} (\alpha_{g,n} \cdot \psi_m^l) \circ_m r_l +$$
$$+ \frac{1}{2} \Bigg(\sum_{i=0}^{l-1} (-1)^{i+1} \left(\sigma_*(\alpha_{g-1,n+2} \cdot \psi_{n+1}^i \psi_{n+2}^{l-1-i}), \eta^{-1} r_l \right) +$$
$$+ \sum_{i+j=l-1} (-1)^{j+1} \left(\rho_*(\alpha_{g_1,|I|+1} \cdot \psi_{|I|+1}^i \otimes \alpha_{g_2,|J|+1} \cdot \psi_{|J|+1}^j), \eta^{-1} r_l \right) \Bigg).$$

The last sum is over all partitions $I \sqcup J = \{1, \ldots, n\}$ and $g_1 + g_2 = g$; the maps σ and ρ identify the points labelled $n+1$ and $n+2$ in the second sum, and the points $|I|+1$ on $\overline{\mathcal{M}}_{g_1+1,|I|+1}$ and $|J|+1$ on $\overline{\mathcal{M}}_{g_2+1,|J|+1}$ in the third sum.

Givental group action on CohFT's

Proposition (Kazarian, Teleman)

The classes

$$\tilde{\alpha}_{g,n} := \left(\exp\left(\sum_{l=1}^{\infty} \widehat{r_l z^l} \right) .\alpha \right)_{g,n}$$

are well-defined cohomology classes with the values in the tensor powers of V that satisfy CohFT constraints; thus, the Givental group acts on cohomological field theories.

Givental stabilisers of TFT's, genus 0

Theorem (D.–Shadrin–Vallette, 2011)

Let $\mathrm{A} = \{\alpha_{0,n}\}_{n\geq 3}$ be a genus 0 topological field theory on a vector space V, making it into a commutative associative algebra. The Lie algebra of the stabiliser of A is spanned by all elements

$$\sum_{p\geq 2} D_p z^{p-1}$$

for which D_p is a differential operator of order at most p on V.

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Sketch of a proof: first, note that for a TFT A, all contributions $\widehat{D_p z^{p-1}}.\mathrm{A}$ live in different cohomological degrees, so we may explore these conditions separately.

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Sketch of a proof: first, note that for a TFT A, all contributions $\widehat{D_p z^{p-1}}.\mathrm{A}$ live in different cohomological degrees, so we may explore these conditions separately. Second, we get rid of cohomology classes: for a sequence of integers $d_0, \ldots, d_n \geq 0$ with $l - 1 + d_0 + d_1 + \cdots + d_n = n - 2$, we compute

$$\int_{\overline{\mathscr{M}}_{0,n+1}} \widehat{D_p z^{p-1}}.\alpha_n \cdot \prod_{m=0}^{n} \psi_m^{d_m}.$$

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$$\int_{\overline{\mathscr{M}}_{0,n+1}} \widehat{D_p z^{p-1}}.\alpha_n \cdot \prod_{m=0}^{n} \psi_m^{d_m}.$$

In fact, this can be rewritten as

$$\langle \tau_{d_0+p-1}\tau_{d_1}\cdots\tau_{d_n}\rangle_0 D_p(f_1 f_2 \cdots f_n)+$$
$$+\sum_{i+j=p-2} (-1)^{j+1} \langle \tau_i \tau_{d_{i_1}} \cdots \tau_{d_{i_r}}\rangle_0 \langle \tau_{d_0}\tau_j \tau_{d_{j_1}} \cdots \tau_{d_{j_s}}\rangle_0 D_p(f_I) f_J +$$
$$+(-1)^p \sum_{m=1}^{n} \langle \tau_{d_0}\tau_{d_1}\cdots\tau_{d_m+p-1}\cdots\tau_{d_n}\rangle_0 f_1 \cdots D_p(f_m)\cdots f_n.$$

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It is known that

$$\langle \tau_{d_0} \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \begin{cases} \binom{n-2}{d_0, \ldots, d_n} & \text{if } d_0 + \cdots + d_n = n-2, \\ 0 & \text{otherwise.} \end{cases}$$

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[Aha! That's where the all those coefficients in

$$\binom{n-2}{d_0+p-1,d_1,\ldots,d_n}D(f_1f_2\cdots f_n)+$$

$$+\sum_{i+j=p-2}(-1)^{j+1}\binom{|I|-2}{i,d_{i_1},\ldots,d_{i_r}}\binom{|J|-1}{j,d_0,d_{j_1},\ldots,d_{j_s}}D(f_I)f_J+$$

$$+(-1)^p\sum_{m=1}^{n}\binom{n-2}{d_0,\ldots,d_m+p-1,\ldots,d_n}f_1\cdots D(f_m)\cdots f_n,$$

are coming from!]

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$$\langle \tau_{d_0}\tau_{d_1}\cdots\tau_{d_n}\rangle_0 = \begin{cases} \binom{n-2}{d_0,\ldots,d_n} & \text{if } d_0+\cdots+d_n = n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Also, for any i, j we have a topological recursion relation

$$\langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_i+1}\cdots\tau_{d_n}\rangle_0 + \langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_j+1}\cdots\tau_{d_n}\rangle_0 = \\ = \sum_{i\in I, j\in J} \langle \tau_{d_{i_1}}\cdots\tau_{d_{i_r}}\tau_0\rangle_0 \langle \tau_0\tau_{d_{j_1}}\cdots\tau_{d_{j_s}}\rangle_0,$$

hinting that we might be able to prove the theorem by a clever induction.

Givental stabilisers of TFT's, genus 0

The actual proof is just a little bit more tricky. First, there is a series of identities generalising

$$D(f_1 f_2 \cdots f_n) = \sum_{1 \leq i < j \leq n} D(f_j f_j) f_1 \cdots \widehat{f_i} \cdots \widehat{f_j} \cdots f_n - \\ - (n-2) \sum_{m=1}^{n} D(f_m) f_1 \cdots \widehat{f_m} \cdots f_n,$$

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(This identity is the identity for $d_0 = n - p - 1$ and $d_1 = \cdots = d_n = 0$ above.)

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(This identity is the identity for $d_0 = n-p-1$ and $d_1 = \cdots = d_n = 0$ above.) All other identities follow by means of topological recursion relations.

Givental stabilisers of TFT's, genus 1

If we want to incorporate the genus 1 information, we can "eliminate" the scalar product, using it to identify $V^{\otimes(n+1)}$ with $\mathrm{Hom}(V^{\otimes n}, V)$ in the genus 0 part, and $V^{\otimes n}$ with $\mathrm{Hom}(V^{\otimes n}, \mathbb{C})$ in the genus 1 part. Main characters: the product $\alpha_{0,3}$ and the "trace" $\alpha_{1,1}$.

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For algebras with trace, the relevant properties of differential operators appear to be Getzler's $1/12$-axiom for order 2:

$$\mathrm{tr}(D_2(f \cdot (-))) = \frac{1}{12} \mathrm{tr}(D_2(f) \cdot (-)),$$

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$$\mathrm{tr}(D_2(f\cdot(-))) = \frac{1}{12}\,\mathrm{tr}(D_2(f)\cdot(-)),$$

and a (new) property of "being compatible with the trace" for order 3 and higher:

$$\mathrm{tr}(\langle f_1, f_2, \ldots, f_{p-1}\rangle_{p-1}^{D_p}\cdot(-)) = 0.$$

Givental stabilisers of TFT's, genus 1

Theorem (D.–Shadrin–Vallette, 2011)

Let $\mathrm{A} = \{\alpha_{0,n}\}_{n\geq 3} \cup \{\alpha_{1,n}\}_{n\geq 1}$ *be a genus* 0 *and* 1 *topological field theory on a vector space* V*, making it into a commutative associative algebra with a trace. The Lie algebra of the stabiliser of* A *only contains elements*

$$\sum_{p\geq 2} D_p z^{p-1}$$

for which D_p *is a differential operator of order at most* p *on* V *that satisfies Getzler's* $1/12$*-axiom for* $p = 2$ *and is compatible with the trace for* $p \geq 3$*.*

Givental stabilisers of TFT's, genus 1

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for which D_p *is a differential operator of order at most* p *on* V *that satisfies Getzler's* $1/12$*-axiom for* $p = 2$ *and is compatible with the trace for* $p \geq 3$*. Under the Gorenstein conjecture for genus* 1*, the Lie algebra of the stabiliser is precisely the linear span of such elements.*

That's all

Thank you for your patience!