Algebra Meets Geometry or Counting with Pictures

Dr Vladimir Dotsenko

School of Mathematics

TCD Open Day
December 1, 2012

What we shall not discuss in this short lecture

(An ingenious tribute to omnipresence of maths, a song “That’s Mathematics” by Tom Lehrer was played.)

Algebra vs Geometry

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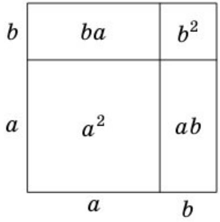

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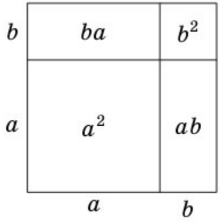
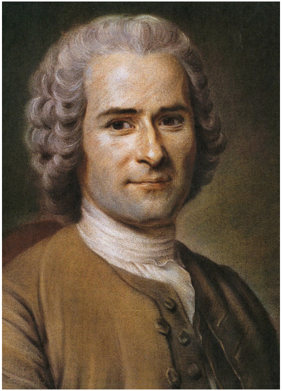


Which explanation do you prefer?

Algebra vs Geometry

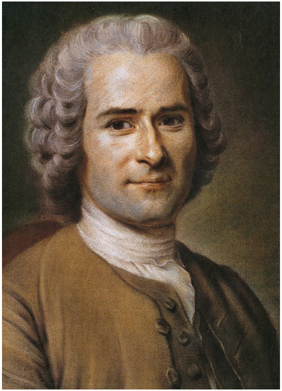
Jean-Jacques Rousseau, *Les Confessions* (1769):

> *La première fois que je trouvai par le calcul que le carré d'un binome était composè du carré de chacune de ses parties et du double produit d'une par autre, malgré la justesse de ma multiplication, je n'en voulus rien croire jusqu'à ce que j'eusse fait la figure.*



Jean-Jacques Rousseau, *Les Confessions* (1769):

> *The first time I found by calculation that the square of a binomial was composed of the square of each its parts added to twice the product of those parts, in spite of the correctness of my multiplication, I would not believe it until I had drawn the figure.*



Adding numbers

$$1+2+\ldots+99+100=?$$

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The primary teacher of Carl Friedrich Gauss (1777–1885) gave this problem to his pupils, hoping that they will spend an hour adding numbers.

Adding numbers

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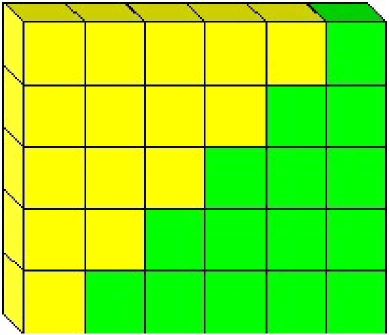
$$\begin{array}{rrrrr} 1 & +2 & +\ldots & +99 & +100 \\ 100 & +99 & +\ldots & +2 & +1 \\ \hline 101 & +101 & \ldots & +101 & +101 \end{array}$$

However, little Gauss came up with a very neat trick:

$$\begin{array}{rrcrr} 1 & +2 & +\ldots & +99 & +100 \\ 100 & +99 & +\ldots & +2 & +1 \\ \hline 101 & +101 & \ldots & +101 & +101 \end{array}$$

so $2S = 101 \cdot 100$, and $S = 5050$.

This approach can be explained geometrically:



we can assemble a rectangular from two triangles made of layers with 1, 2, 3 etc. boxes.

Adding odd numbers

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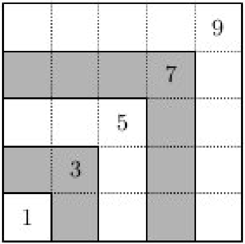
The trick of Gauss works:

$$\begin{array}{rrrrr} 1 & +3 & +\ldots & +97 & +99 \\ 99 & +97 & +\ldots & +3 & +1 \\ \hline 100 & 100 & \ldots & 100 & 100 \end{array}$$

so $2S = 50 \cdot 100$, and $S = 2500 = 50^2$.

Adding odd numbers

Alternatively, we can look at the square cut into "corner layers"



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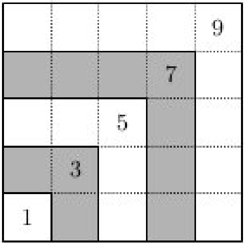


and count the boxes in two different ways.

Adding squares

$$1 + 4 + 9 + \ldots + 100^2 = ?$$

$$1+4+9+\ldots+100^2=?$$

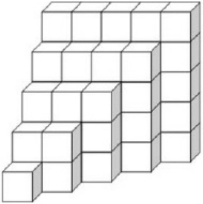
This one is more tricky. We can solve it by assembling more interesting geometric shapes.

Adding squares

A pyramid in History Museum, Strasbourg. Each layer is a square made of canonballs.

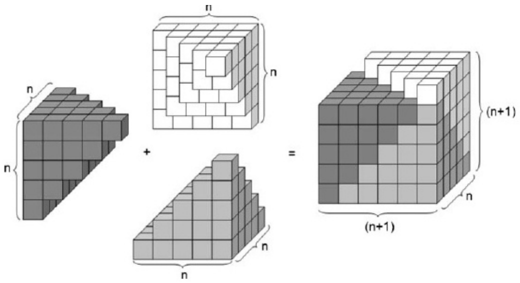
Adding squares

We shall use six pyramids made in a similar way:



Adding squares

Those six pyramids can be assembled into a rectangular box made of two almost rectangular shapes:



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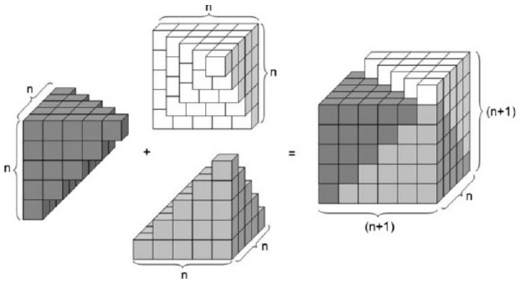

Overall,

$$6(1 + 4 + 9 + \ldots + n^2) = n(n+1)(2n+1).$$

Integrating?

Suppose we want to compute the area under the parabola $y = x^2$ for $0 \le x \le 1$.

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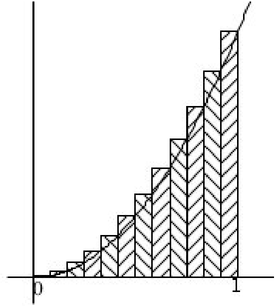

We approximate that area by many, say N, rectangles whose areas are

$$\frac{1}{N} \cdot \left(\frac{1}{N}\right)^2, \frac{1}{N} \cdot \left(\frac{2}{N}\right)^2, \ldots, \frac{1}{N} \cdot \left(\frac{N}{N}\right)^2$$

Integrating?

The total area of our rectangles is

$$\frac{1}{N}\cdot\left(\frac{1}{N}\right)^2+\frac{1}{N}\cdot\left(\frac{2}{N}\right)^2+\ldots+\frac{1}{N}\cdot\left(\frac{N}{N}\right)^2=$$

$$=\frac{1+4+\ldots+N^2}{N^3}=\frac{N(N+1)(2N+1)}{6\cdot N^3}=$$

$$=\frac{1}{6}\left(1+\frac{1}{N}\right)\left(2+\frac{1}{N}\right)\approx\frac{1}{3}.$$

Integrating?

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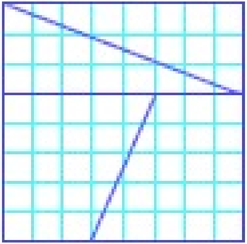
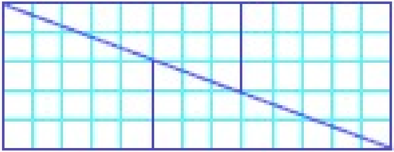
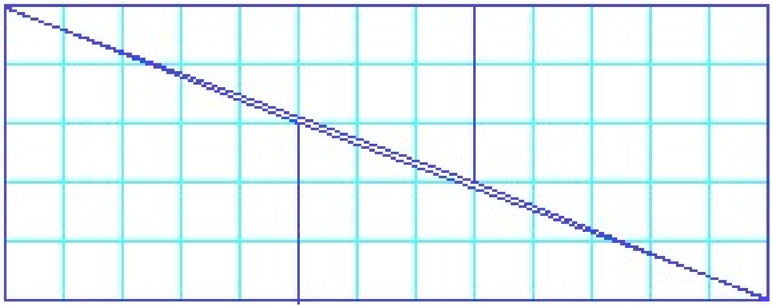
In mathematics, it is quite common to exhibit “intelligent laziness”, identifying similar problems, and solving each problem just once.

WARNING: $64 = 65$??





Mystery revealed: $64 = 65 - 1$!!



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Turns out that all example of this sort have empty space of area exactly 1!

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Let us say that a parallelogram on grid paper is “empty” if the only grid nodes it covers are its vertices. The area of every empty parallelogram is exactly equal to 1, regardless of how thin it is.

That's all folks!

Thank you for your attention!