MA1S11 (Dotsenko) Solutions to Tutorial/Exercise Sheet 8

Week 10, Michaelmas 2013

1. Investigate fully the rational function $f(x) = \frac{x^2}{1-x^3}$, and sketch its graph.

   Solution. The graph does not possess any of the usual symmetries. The only x-intercept is at 0, the y-intercept is at $f(0) = 0$. The only vertical asymptote is $x = 1$.

   The points 0, and 1 divide the x-axis into the intervals $(-\infty, 0)$, $(0, 1)$, $(1, +\infty)$. Signs of the corresponding factors result in the following signs for f:

interval	$(-\infty, 0)$	$(0, 1)$	$(1, +\infty)$
signs of factors $1 - x^3$, x	$(+)(-)$	$(+)(+)$	$(-)(+)$
sign of f	$-$	$+$	$-$

The limiting behaviour at infinity is given by

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{-x + \frac{1}{x^2}} = 0,$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{-x + \frac{1}{x^2}} = 0,$$

therefore the x-axis is a horisontal asymptote of the graph.

The first derivative of f is

$$f'(x) = \frac{2x \cdot (1 - x^3) - x^2(-3x^2)}{(1 - x^3)^2} = \frac{x^4 + 2x}{(1 - x^3)^2} = \frac{x(x^3 + 2)}{(1 - x^3)^2}.$$

Its roots are $x = 0$ and $x = \sqrt[3]{-2} \approx -1.2599$.

Signs of the corresponding factors result in the following signs for f':

interval	$(-\infty, \sqrt[3]{-2})$	$(\sqrt[3]{-2}, 0)$	$(0, 1)$	$(1, +\infty)$
signs of factors x, $x^3 + 2$, and $(1 - x^3)^2$	$(-)(-)(+)$	$(-)(+)(+)$	$(+)(+)(+)$	$(+)(+)(+)$
sign of f'	$+$	$-$	$+$	$+$

This suggests that f is increasing on $(-\infty, -\sqrt[3]{-2}]$, on $[0, 1)$, and on $(1, +\infty)$, and is decreasing on $[-\sqrt[3]{2}, 0]$. Therefore, there is a relative maximum at $x = -\sqrt[3]{2}$ and a relative minimum at $x = 0$.

The second derivative of f is

$$f''(x) = \frac{(4x^3+2)\cdot(1-x^3)^2 - (x^4+2x)2(1-x^3)(-3x^2)}{(1-x^3)^4} =$$
$$= \frac{(4x^3+2)(1-x^3)+(x^4+2x)(6x^2)}{(1-x^3)^3} = 2\frac{x^6+7x^3+1}{(1-x^3)^3}.$$

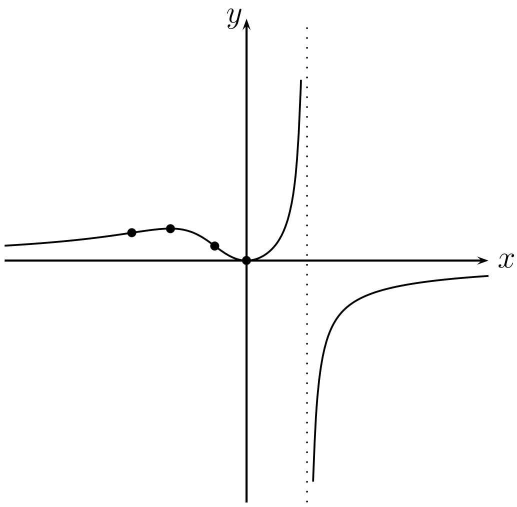
To find its roots, we denote $x^3 = t$, and solve the equation $t^2 + 7t + 1 = 0$, obtaining $t_1 = \frac{-7+\sqrt{45}}{2}$ and $t_2 = \frac{-7-\sqrt{45}}{2}$, which approximately are -0.1459 and -6.8541. Recalling that $t = x^3$, we obtain the following approximate values for x: $x_1 \approx -0.5264$ and $x_2 \approx -1.8995$.

Signs of the corresponding factors result in the following signs for f'':

interval	$(-\infty, \sqrt[3]{t_2})$	$(\sqrt[3]{t_2}, \sqrt[3]{t_1})$	$(\sqrt[3]{t_1}, 1)$	$(1, +\infty)$
signs of factors (x^3-t_1), (x^3-t_2), and $(1-x^3)^3$	$(-)(-)(+)$	$(-)(+)(+)$	$(+)(+)(+)$	$(+)(+)(-)$
sign of f''	$+$	$-$	$+$	$-$

This suggests that f is concave up on $(-\infty, x_2)$ and on $(x_1, 1)$, and is concave down on (x_2, x_1) and on $(1, +\infty)$. Both x_1 and x_2 are inflection points.

Based on these computations, we sketch a graph as follows:



2. Determine the relative and the absolute extrema of the function f on the closed interval $[-2, 3]$, if

$$f(x) = -\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 + 1.$$

Solution. This function is differentiable everywhere on the open interval $(-2, 3)$. Its derivatives are

$$\begin{aligned} f'(x) &= -x^3 + x^2 + 2x = -x(x-2)(x+1), \\ f''(x) &= -3x^2 + 2x + 2. \end{aligned}$$

Solving $f'(x) = 0$ we get $x = 0, 2, -1$ as candidates for relative extrema. Applying the second derivative test we see that

$$\begin{aligned} f''(-1) &= -3 - 2 + 2 = -3 < 0, \\ f''(0) &= 2 > 0, \\ f''(2) &= -12 + 4 + 2 = -6 < 0. \end{aligned}$$

Therefore, -1 and 2 are relative maxima and 0 is a relative minimum. The values of f at the relative extrema are

$$f(0) = 1, \qquad f(2) = \frac{11}{3}, \qquad f(-1) = \frac{17}{12} \tag{1}$$

This is to be compared with the values of f at the endpoints $-2, 3$:

$$f(-2) = -\frac{5}{3}, \qquad f(3) = -\frac{5}{4}. \tag{2}$$

We conclude that f has an absolute maximum at $x = 2.$ and an absolute minimum at the left endpoint $x = -2$.