1S11: Calculus for students in Science

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TCD

Lecture 10

Limits at infinity

A function f is said to *have the limit L as x tends to $+\infty$* if the values $f(x)$ get as close as we like to L as x increases without bound. In this case, one writes

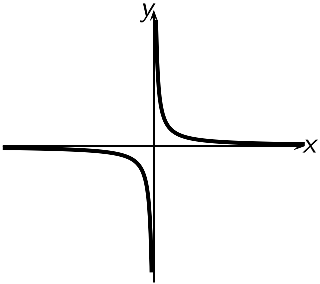
$$\lim_{x\to+\infty} f(x) = L.$$

Similarly, a function f is said to *have the limit L as x tends to $-\infty$* if the values $f(x)$ get as close as we like to L as x decreases without bound. In this case, one writes

$$\lim_{x\to-\infty} f(x) = L.$$

Limits at infinity

For example, let us consider the function $f(x) = 1/x$:

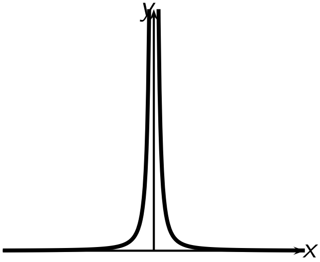


In this case, we have

$$\lim_{x\to+\infty} f(x) = 0 \text{ and } \lim_{x\to-\infty} f(x) = 0.$$

Limits at infinity

Now, let us consider the function $f(x) = 1/x^2$:



In this case, we also have

$$\lim_{x\to+\infty} f(x) = 0 \text{ and } \lim_{x\to-\infty} f(x) = 0.$$

Limits at infinity

In terms of limits at infinity we can make the notion of a horisontal asymptote more precise:

The graph of a function f has the line $y = L$ as a horisontal asymptote if at least one of the two following situations occur:

$$\lim_{x\to+\infty} f(x) = L, \lim_{x\to-\infty} f(x) = L.$$

E.g., both for $f(x) = 1/x$ and $f(x) = 1/x^2$, both of the formulas apply when $L = 0$.

Exercise. Sketch examples of graphs for which exactly one of those situations occurs.

Infinite limits at infinity

A function f is said to *have the limit* $+\infty$ *as* x *tends to* $+\infty$ *(or* $-\infty$*)* if the values $f(x)$ *all* increase without bound as x increases without bound (decreases without bound). In this case, one writes

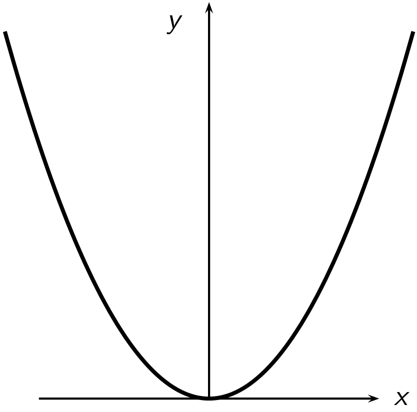
$$\lim_{x\to+\infty} f(x) = +\infty \qquad (\lim_{x\to-\infty} f(x) = +\infty).$$

A function f is said to *have the limit* $-\infty$ *as* x *tends to* $+\infty$ *(or* $-\infty$*)* if the values $f(x)$ *all* decrease without bound as x increases without bound (decreases without bound). In this case, one writes

$$\lim_{x\to+\infty} f(x) = -\infty \qquad (\lim_{x\to-\infty} f(x) = -\infty).$$

Infinite limits at infinity

For example, let us consider the function $f(x) = x^2$:

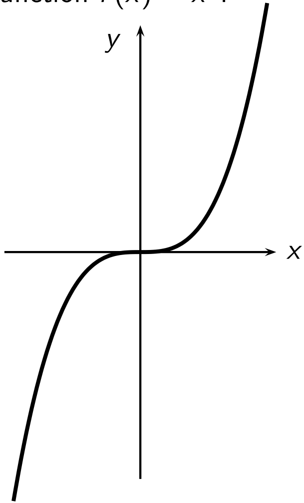


In this case, we have

$$\lim_{x\to+\infty} f(x) = +\infty \text{ and } \lim_{x\to-\infty} f(x) = +\infty.$$

Infinite limits at infinity

Now, let us consider the function $f(x) = x^3$:



In this case, we have

$$\lim_{x \to +\infty} f(x) = +\infty \text{ and } \lim_{x \to -\infty} f(x) = -\infty.$$

Limits of polynomials at infinity

Theorem. Suppose that $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_n \neq 0$. Then

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(a_n x^n),$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(a_n x^n).$$

Proof. We have

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n = a_n x^n \left(\frac{a_0}{a_n x^n} + \frac{a_1}{a_n x^{n-1}} + \cdots + \frac{a_{n-1}}{a_n x} + 1\right),$$

and the expression in the brackets clearly has the limit 1 as $x \to +\infty$ or $x \to -\infty$, since all terms except for 1 have the limit 0.

Informally, this theorem says that *the limiting behaviour at infinity of a polynomial exactly matches the behaviour of its highest degree term*.

Limits of rational functions at infinity

Theorem. Suppose that $f(x) = \frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_mx^m}$, where $a_n, b_m \neq 0$. Then

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{a_nx^n}{b_mx^m},$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{a_nx^n}{b_mx^m}.$$

Proof. We have

$$f(x) = \frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_mx^m} = \frac{a_nx^n}{b_mx^m}\cdot\frac{\frac{a_0}{a_nx^n}+\frac{a_1}{a_nx^{n-1}}+\cdots+\frac{a_{n-1}}{a_nx}+1}{\frac{b_0}{b_mx^m}+\frac{b_1}{b_mx^{m-1}}+\cdots+\frac{b_{m-1}}{b_mx}+1},$$

and both the numerator and the denominator of the second fraction clearly have the limit 1 as $x \to +\infty$ or $x \to -\infty$, since all terms except for 1 have the limit 0.

Informally, this theorem says that *the limiting behaviour at infinity of a rational function exactly matches the behaviour of the ratio of highest degree terms in the numerator and the denominator*.

Limits of rational functions at infinity: examples

Example 1. We have

$$\lim_{x\to+\infty}\frac{3x+5}{6x-8}=\lim_{x\to+\infty}\frac{3x}{6x}\cdot\frac{1+\frac{5}{3x}}{1-\frac{8}{6x}}=\lim_{x\to+\infty}\frac{3x}{6x}=\frac{1}{2}.$$

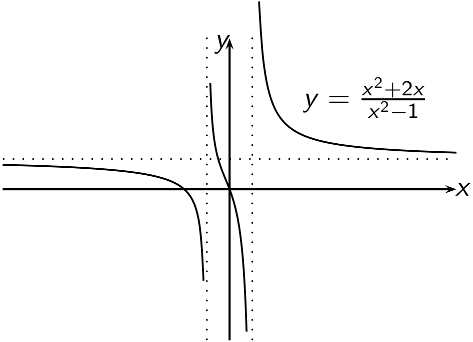
Example 2. We have

$$\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}=\lim_{x\to-\infty}\frac{4x^2}{2x^3}\cdot\frac{1-\frac{1}{4x}}{1-\frac{5}{2x^3}}=\lim_{x\to-\infty}\frac{4x^2}{2x^3}=\lim_{x\to-\infty}\frac{2}{x}=0.$$

Example 3. We have

$$\lim_{x\to+\infty}\frac{5x^3-2x^2+1}{1-3x}=\lim_{x\to+\infty}\frac{5x^3}{-3x}\cdot\frac{1-\frac{2}{5x}+\frac{1}{5x^3}}{1-\frac{1}{3x}}=\lim_{x\to+\infty}\frac{5x^2}{-3}=-\infty.$$

One example from Lecture 5

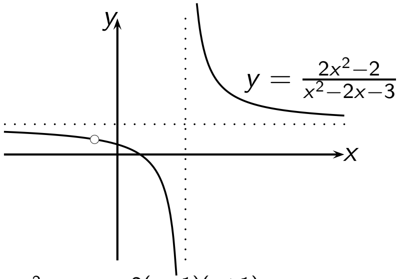


The graph of $f(x) = \frac{x^2+2x}{x^2-1} = \frac{x^2+2x}{(x-1)(x+1)}$ has vertical asymptotes $x = 1$ and $x = -1$, and a horisontal asymptote $y = 1$. Now we know which limits control those asymptotes:

$$\lim_{x\to 1^+} f(x) = +\infty, \lim_{x\to 1^-} f(x) = -\infty, \lim_{x\to -1^+} f(x) = +\infty, \lim_{x\to -1^-} f(x) = -\infty$$

for the vertical asymptotes, and $\lim_{x\to+\infty} f(x) = 1$, $\lim_{x\to-\infty} f(x) = 1$ for the horisontal asymptote.

Another example from Lecture 5



The graph of $f(x) = \frac{2x^2-2}{x^2-2x-3} = \frac{2(x-1)(x+1)}{(x+1)(x-3)}$ has a vertical asymptote $x = 3$, a horisontal asymptote $y = 2$, and also the point $(-1, 1)$ which it approaches both on the left and on the right but does not touch. This corresponds to the existence of the limits

$$\lim_{x\to 3^+} f(x) = +\infty, \lim_{x\to 3^-} f(x) = -\infty,$$

$$\lim_{x\to +\infty} f(x) = 2, \lim_{x\to -\infty} f(x) = 2,$$

$$\lim_{x\to -1} f(x) = 1.$$

More difficult limits at infinity

For ratios involving square roots, the same method we used before is useful:

$$\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to+\infty}\frac{\sqrt{x^2(1+\frac{2}{x^2})}}{3x(1-\frac{2}{x})}=\lim_{x\to+\infty}\frac{|x|}{3x}=\frac{1}{3},$$

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to-\infty}\frac{\sqrt{x^2(1+\frac{2}{x^2})}}{3x(1-\frac{2}{x})}=\lim_{x\to-\infty}\frac{|x|}{3x}=-\frac{1}{3}.$$

More difficult limits at infinity

For differences $\sqrt{f(x)} - \sqrt{g(x)}$, or simply $\sqrt{f(x)} - h(x)$, it is useful to apply the formula $a - b = \frac{a^2-b^2}{a+b}$:

$$\lim_{x\to+\infty}(\sqrt{x^6+5} - x^3) = \lim_{x\to+\infty} \frac{(\sqrt{x^6+5})^2 - (x^3)^2}{\sqrt{x^6+5}+x^3} =$$

$$= \lim_{x\to+\infty} \frac{5}{\sqrt{x^6+5}+x^3} = 0,$$

$$\lim_{x\to+\infty}(\sqrt{x^6+5x^3} - x^3) = \lim_{x\to+\infty} \frac{(\sqrt{x^6+5x^3})^2 - (x^3)^2}{\sqrt{x^6+5x^3}+x^3} =$$

$$= \lim_{x\to+\infty} \frac{5x^3}{\sqrt{x^6+5x^3}+x^3},$$

and noticing that for large positive x we can use our previous method and write $\sqrt{x^6+5x^3} + x^3 = x^3\left(\sqrt{1+\frac{5}{x^3}}+1\right)$, we finally conclude that $\lim_{x\to+\infty}(\sqrt{x^6+5x^3} - x^3) = \frac{5}{2}$.