1S11: Calculus for students in Science

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TCD

Lecture 11

Solution to the exercise from yesterday

Reminder. The graph of a function f has the line $y = L$ as a horisontal asymptote if at least one of the two following situations occur:

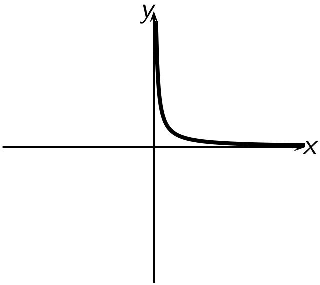
$$\lim_{x \to +\infty} f(x) = L, \lim_{x \to -\infty} f(x) = L.$$

E.g., both for $f(x) = 1/x$ and $f(x) = 1/x^2$, both of the formulas apply when $L = 0$.

Exercise. Sketch examples of graphs for which exactly one of those situations occurs.

Solution to the exercise from yesterday

For example, let us consider the function $f(x) = 1/x, x > 0$:



In this case, we have

$$\lim_{x \to +\infty} f(x) = 0,$$

but there is no limit for $x \to -\infty$ since there are no values!

More difficult limits at infinity: reminder

For differences $\sqrt{f(x)} - \sqrt{g(x)}$, or simply $\sqrt{f(x)} - h(x)$, it is useful to apply the formula $a - b = \frac{a^2-b^2}{a+b}$:

$$\lim_{x\to+\infty}(\sqrt{x^6+5} - x^3) = \lim_{x\to+\infty} \frac{(\sqrt{x^6+5})^2 - (x^3)^2}{\sqrt{x^6+5}+x^3} =$$

$$= \lim_{x\to+\infty} \frac{5}{\sqrt{x^6+5}+x^3} = 0,$$

$$\lim_{x\to+\infty}(\sqrt{x^6+5x^3} - x^3) = \lim_{x\to+\infty} \frac{(\sqrt{x^6+5x^3})^2 - (x^3)^2}{\sqrt{x^6+5x^3}+x^3} =$$

$$= \lim_{x\to+\infty} \frac{5x^3}{\sqrt{x^6+5x^3}+x^3},$$

and noticing that for large positive x we can use our previous method and write $\sqrt{x^6+5x^3}+x^3 = x^3\left(\sqrt{1+\frac{5}{x^3}}+1\right)$, we finally conclude that $\lim_{x\to+\infty}(\sqrt{x^6+5x^3} - x^3) = \frac{5}{2}$.

Trigonometric functions at infinity

Since $\sin x$ and $\cos x$ oscillate between -1 and 1 as $x \to +\infty$ or $x \to -\infty$, neither of these functions has a limit at infinity.

Let us note, however, that it does not preclude limits like $\lim\limits_{x\to+\infty} \frac{\sin x}{x}$ from having a defined value. Indeed, as $x \to +\infty$, the value of $\sin x$ is between -1 and 1, and the value of x increases without bound, so the ratio of these quantities has the limit 0:

$$\lim_{x\to+\infty} \frac{\sin x}{x} = 0.$$

On a lighter note, please do **not** do anything like that:

$$\lim_{x\to+\infty} \frac{\sin x}{x} = \lim_{x\to+\infty} \frac{\sin \cancel{x}}{\cancel{x}} = \lim_{x\to+\infty} \frac{\sin}{1} = \sin\,.$$

Continuity

Definition. A function $f(x)$ (defined on an open interval containing $x = c$) is said to be *continuous* at the point $x = c$ if

- $f(c)$ is defined;
- $\lim\limits_{x\to c} f(x)$ exists (as a finite number);
- $\lim\limits_{x\to c} f(x) = f(c)$.

Example. Let us consider the functions

$$f(x) = \frac{x^2-4}{x-2}, \quad g(x) = \begin{cases} \frac{x^2-4}{x-2}, x \neq 2, \\ 3, x = 2, \end{cases} \qquad h(x) = \begin{cases} \frac{x^2-4}{x-2}, x \neq 2, \\ 4, x = 2. \end{cases}$$

These functions all coincide for $x \neq 2$, so they have the same limit at $x \to 2$: $\lim\limits_{x\to 2} \frac{x^2-4}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$. The first function is undefined for $x = 2$, so it is not continuous, the second function is defined, but the values do not match, so it is not continuous either, and the third function is continuous.

Continuity on an interval

We shall say that a function f is continuous on an open interval (a, b) if it is continuous at each point of that interval. For close interval, the definition involves one-sided limits:

Definition. A function $f(x)$ is said to be *continuous from the left* at the point $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$ (so that the limit is defined, the value is defined, and they are equal). A function $f(x)$ is said to be *continuous from the right* at the point $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$ (so that the limit is defined, the value is defined, and they are equal).

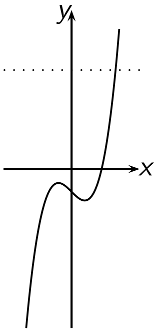
We shall say that a function f is continuous on a closed interval $[a, b]$ if

- f is continuous on (a, b);
- f is continuous from the right at $x = a$;
- f is continuous from the left at $x = b$.

Sneak peek: what is continuity good for?

Theorem. (*Intermediate Value Theorem*) If a function f is continuous on the closed interval $[a, b]$, for every value k between $f(a)$ and $f(b)$ the equation $f(x) = k$ has a solution on $[a, b]$.

Example.



For $f(x) = x^3 - x - 1$, we have $f(-2) = -7$, and $f(2.1) = 6.161$, so on the interval $[-2, 2.1]$ this function assumes any value in between, e.g. $\sqrt{7} + \sqrt[3]{5} \approx 4.356$.

Continuity on an interval: examples

Example. The function $f(x) = \frac{1}{x}$ is continuous on the closed interval $[2, 3]$. The same function is also continuous on the open interval $(0, 1)$: for every c between 0 and 1 the value $\frac{1}{c}$ is defined, and $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$. However, the same function is not continuous on the closed interval $[0, 1]$ since $f(0)$ is not defined.

Example. The function $f(x) = \sqrt{9 - x^2}$ is continuous on the closed interval $[-3, 3]$ (its natural domain), since we have

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{9 - x^2} = 0 = f(3),$$
$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9 - x^2} = 0 = f(-3).$$

Continuity and arithmetics

Properties of limits immediately imply the following theorem:

Theorem. If functions f and g are continuous at c, then

- $f + g$ is continuous at c;
- $f - g$ is continuous at c;
- fg is continuous at c;
- f/g is continuous at c if $g(c) \neq 0$, and is not continuous at c if $g(c) = 0$.

A useful remark. The function $f(x) = |x|$ is continuous everywhere. Indeed, for $c > 0$ we have $\lim\limits_{x\to c} |x| = \lim\limits_{x\to c} x = c = |c|$, for $c < 0$ we have $\lim\limits_{x\to c} |x| = \lim\limits_{x\to c} (-x) = -c = |c|$, and for $c = 0$ we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0, \text{ and } \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0,$$

so the two-sided limit exists and is equal to 0.

Continuity and arithmetics

Theorem.

1. A polynomial is continuous at all points on the real line.
2. A rational function is continuous at all points where the denominator is not equal to zero.

Proof. We already know that a polynomial $p(x)$ has the limit $p(a)$ at each $x = a$, and a rational function $f(x) = \frac{p(x)}{q(x)}$ has the limit $f(a) = \frac{p(a)}{q(a)}$ at each point a where it is defined (that is where $q(a) \neq 0$), so there is nothing to prove.

Example. The function $f(x) = \frac{2x^2-2}{x^2-2x-3}$ we discussed several times is continuous at all points where $x^2 - 2x - 3 \neq 0$, that is x different from $x = -1$ and $x = 3$. Even though this function has a well defined two-sided limit at $x = -1$, it is not defined at that point and so is not continuous (although in such a situation, a function is said to have a "removable discontinuity").

Continuity and composition

For limits, we only discussed their relationship with arithmetics. It turns out that behaviour with respect to composition is better expressed through the notion of continuity:

Theorem. Suppose that the function g has the limit L as x approaches c, $\lim\limits_{x\to c} g(x) = L$. Suppose also that the function f is continuous at L. In that case, we have $\lim\limits_{x\to c} f(g(x)) = f(L)$. In other words,

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right).$$

The same statement applies if $\lim\limits_{x\to c}$ is replaced everywhere by one of the limits $\lim\limits_{x\to c^-}$, $\lim\limits_{x\to c^+}$, $\lim\limits_{x\to+\infty}$, $\lim\limits_{x\to-\infty}$.

Theorem.

1. If a function g is continuous at c, and the function f is continuous at $g(c)$, then the function $f\circ g$ is continuous at c.
2. If both functions f and g are continuous everywhere on the real line, then their composition $f\circ g$ is continuous everywhere.

Intermediate Value Theorem

Theorem. (*Intermediate Value Theorem*) If a function f is continuous on the closed interval $[a, b]$, for every value k between $f(a)$ and $f(b)$ the equation $f(x) = k$ has a solution on $[a, b]$.

Application 1. A particular case of the Intermediate Value Theorem applies for $k = 0$, that is for the equation $f(x) = 0$. In this case it states that if a function f is continuous on $[a, b]$ and assumes nonzero values of opposite signs at a and b, then there is a solution of $f(x) = 0$ on (a, b). We shall later learn some methods of approaching solving equations based on calculus.

Application 2. The number $\sqrt{2}$ exists. Indeed, the function $f(x) = x^2$ is continuous, and we have $f(1) = 1 < 2 < 4 = f(2)$. Therefore, $f(x) = 2$ for some x between 1 and 2.

Application 3. Whichever shape on the plane we take, there is a straight line cutting it into two parts of equal areas. Moreover, whichever two shapes in 3d space we take, there is a plane cutting each of them into two parts of the same volume. (For obvious reasons, the latter result is called "Ham sandwich theorem".)