1S11: Calculus for students in Science

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TCD

Lecture 12

An important announcement

There will be no calculus lecture on Monday October 21 at 4pm (Monday linear algebra lecture, and all lectures from Tuesday on will take place as planned).

Make sure you use that free time for revision. In particular, look at the exercises after sections 1.3 and 1.5 of the textbook. If you do not have the book, find someone who has, and make xerox copies of exercise pages (3 pages for each chapter). Getting enough practice with these exercises is important for both your exam performance and being comfortable with the remaining part of the course.

Continuity of inverse functions

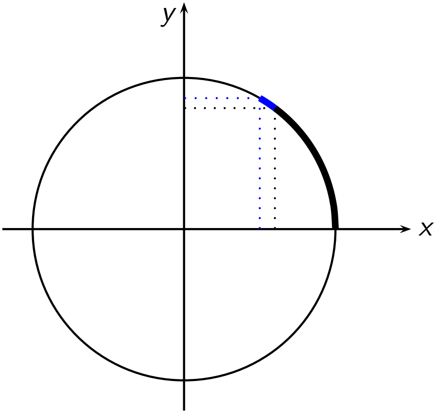
Theorem. Suppose that f is a one-to-one function that is continuous at each point of its domain. Then the inverse f^{-1} is continuous at each point of its domain, that is f^{-1} is continuous at every point of the range of f.

Example 1. Since $f(x) = x^3$ is a polynomial and is therefore continuous everywhere, its inverse function $g(x) = \sqrt[3]{x}$ is also continuous everywhere.

Example 2. The function $f(x) = x^3 + x$ is a polynomial and is therefore continuous everywhere. Also, it is the sum of two increasing functions, so it is increasing, and therefore is one-to-one. It follows that this function has a continuous inverse. Note that in this case, unlike the case of x^3, it is not easy to describe the inverse function by a formula, but the theorem still guarantees its continuity!

Trigonometric functions revisited

We already discussed the fact that trigonometric functions do not really have any defined limit at infinity. However, the limiting behaviour of those at finite points is good: these functions are continuous everywhere.



Informally: a small change of the point on the circle results in small changes in its x- and y-coordinates.

Trigonometric functions revisited

Theorem. Every trigonometric function is continuous at all points of its natural domain:

$$\lim_{x\to c}\sin x=\sin c,\quad \lim_{x\to c}\cos x=\cos c,\quad \lim_{x\to c}\tan x=\tan c,$$
$$\lim_{x\to c}\csc x=\csc c,\quad \lim_{x\to c}\sec x=\sec c,\quad \lim_{x\to c}\cot x=\cot c.$$

Example. The limit $\lim\limits_{x\to 1}\cos\left(\frac{x^2-1}{x-1}\right)$ can be evaluated from the principles we already know: since cos is a continuous function, we have

$$\lim_{x\to 1}\cos(g(x))=\cos\lim_{x\to 1}g(x)$$

whenever $\lim\limits_{x\to 1}g(x)$ exists.Therefore,

$$\lim_{x\to 1}\cos\left(\frac{x^2-1}{x-1}\right)=\lim_{x\to 1}\cos\left(x+1\right)=\cos\lim_{x\to 1}(x+1)=\cos 2.$$

Trigonometric functions revisited

One of the most important limit for applications of calculus is $\lim_{x \to 0} \frac{\sin x}{x}$. So far we have not proved any results that would allow to approach this limit.

Theorem. $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

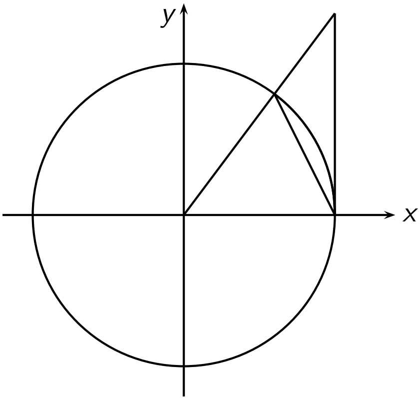
Informal proof. The key idea of the proof is very simple but very important. Suppose that we have three functions $f(x)$, $g(x)$, and $h(x)$, and that we can prove the inequalities

$$g(x) \leq f(x) \leq h(x)$$

for all x in some open interval containing the number c, possibly with the exception of c itself. Then if g and h have the same limit L at c, then f also has the limit L at c. This theorem is often called "The Squeezing Theorem": the values of f are "squeezed" between values of g and h. In Russian maths folklore, they call it "The Two Policemen Theorem": a suspect that is taken to the police station guarded by two policemen on the left and on the right will not be able to escape.

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

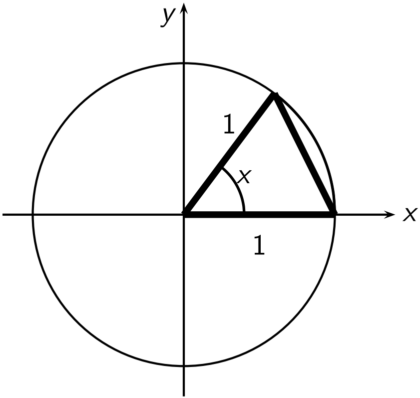
We shall apply the Squeezing Theorem for $f(x) = \frac{\sin x}{x}$, $g(x) = \cos x$, and $h(x) = 1$, on $(-\pi/2, \pi/2)$. Why $\cos x \leq \frac{\sin x}{x} \leq 1$? It is enough to prove it for $(0, \pi/2)$ since the functions involved are even. On that interval, it is the same as $\sin x \leq x \leq \tan x$. Finally, we examine the geometric picture



where we can actually find all the quantities involved!

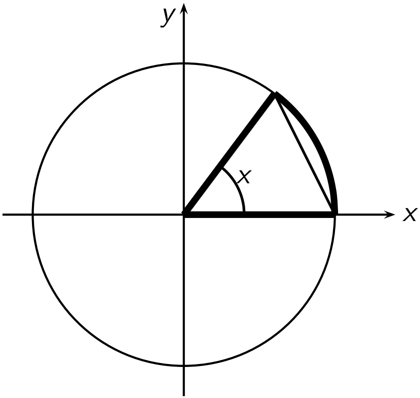
$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Indeed,



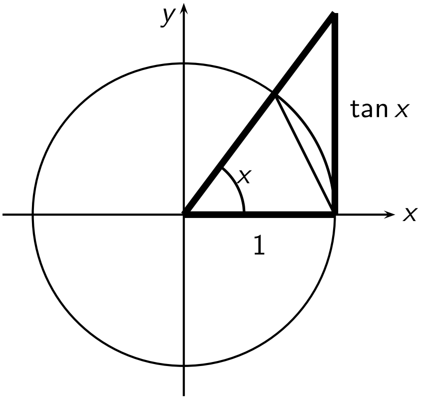
the area of the small triangle is $\frac{1}{2} \cdot 1 \cdot 1 \cdot \sin x = \frac{1}{2} \sin x$,

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$



the area of the sector is $\pi \cdot \frac{x}{2\pi} = \frac{1}{2}x$,

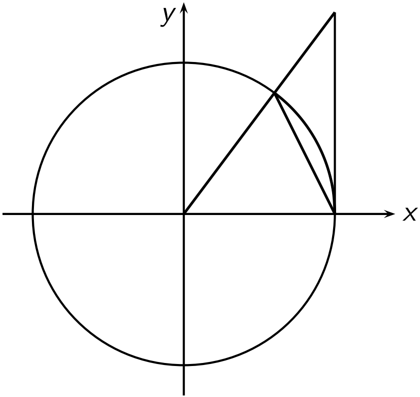
$\lim_{x \to 0} \frac{\sin x}{x} = 1$



and the area of the large triangle is $\frac{1}{2} \cdot 1 \cdot \tan x = \frac{1}{2} \tan x$.

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Therefore an obvious inequality between the areas



implies

$$\sin x \leq x \leq \tan x,$$

which is what we needed.

$\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

Finally,

$$\sin x \leq x \leq \tan x$$

guarantees that for $0 < x < \pi/2$ we have

$$\cos x \leq \frac{\sin x}{x} \leq 1,$$

so since our functions are even, we conclude that for $x \neq 0$ in $(-\pi/2, \pi/2)$ we have

$$\cos x \leq \frac{\sin x}{x} \leq 1,$$

so by the Squeezing Theorem (since $\lim\limits_{x \to 0} \cos x = 1$) we conclude that $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

Consequences of $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

$\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{1-(\cos x)^2}{(1+\cos x)x^2}=\lim\limits_{x\to 0}\frac{(\sin x)^2}{(1+\cos x)x^2}=\lim\limits_{x\to 0}\left[\left(\frac{\sin x}{x}\right)^2\frac{1}{1+\cos x}\right]$,
so $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^2\lim\limits_{x\to 0}\frac{1}{1+\cos x}=1^2\cdot\frac{1}{2}=\frac{1}{2}$.
$\lim\limits_{x\to 0}\frac{1-\cos x}{x}=\lim\limits_{x\to 0}\left[\frac{1-\cos x}{x^2}\cdot x\right]=\frac{1}{2}\cdot 0=0$.
$\lim\limits_{x\to 0^+}\frac{\sin x}{x^2}=\lim\limits_{x\to 0^+}\left[\frac{\sin x}{x}\cdot\frac{1}{x}\right]=+\infty$.
$\lim\limits_{x\to 0}\frac{\tan x}{x}=\lim\limits_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right]=1\cdot 1=1$.
$\lim\limits_{x\to 0}\frac{\sin 2x}{x}=\lim\limits_{x\to 0}2\frac{\sin 2x}{2x}=\lim\limits_{2x=t\to 0}2\frac{\sin t}{t}=2$.
$\lim\limits_{x\to 0}\frac{2-\cos 3x-\cos 4x}{x}=\lim\limits_{x\to 0}\left[\frac{1-\cos 3x}{x}+\frac{1-\cos 4x}{x}\right]=0$.
$\lim\limits_{x\to 0}\frac{\sin 3x}{\sin 5x}=\lim\limits_{x\to 0}\frac{3\frac{\sin 3x}{3x}}{5\frac{\sin 5x}{5x}}=\frac{3}{5}$.

Informal consequences of $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

1. $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ means informally that for small x we have $\sin x\approx x$,
2. $\lim\limits_{x\to 0}\frac{\tan x}{x}=1$ means informally that for small x we also have $\tan x\approx x$,
3. $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$ means informally that for small x we have $\cos x\approx 1-\frac{x^2}{2}$.

These approximate formulas give examples of a general strategy of differential calculus: replacing a function by a polynomial expression that approximates it very well for small x (or x close to the given point a). Our next goal will be finding the same sort of approximations for more complicated functions, and learning general rules for that.