1S11: Calculus for students in Science

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TCD

Lecture 14

1637: René Descartes publishes “La Géométrie”, that is “Geometry”, a book which did not itself address calculus, but however changed once and forever the way we relate geometric shapes to algebraic equations. Later works of creators of calculus definitely relied on Descartes’ methodology and revolutionary system of notation in the most fundamental way.

René Descartes (1596–1660), courtesy of Wikipedia

1638: In a letter to Mersenne, Pierre de Fermat explains what later becomes the key point of his works "Methodus ad disquirendam maximam et minima" and "De tangentibus linearum curvarum", that is "Method of discovery of maximums and minimums" and "Tangents of curved lines" (published posthumously in 1679). This is not differential calculus *per se*, but something equivalent.

Pierre de Fermat (1601–1665), courtesy of Wikipedia

Some historical notes on calculus

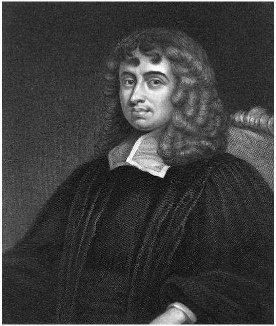
In 1650s, slightly younger scientists like Blaise Pascal and Christiaan Huygens also used methods similar to those of Fermat to study maximal and minimal values of functions. They did not talk about anything like limits, but in fact were doing exactly the things we do in modern calculus for purposes of geometry and optics.

Christiaan Huygens (1629–1695), courtesy of Wikipedia

Blaise Pascal (1623–1662), courtesy of Wikipedia

Some historical notes on calculus

1669-70: Isaac Barrow publishes "Lectiones Opticae" and "Lectiones Geometricae" ("Lectures on Optics" and "Lectures on Geometry"), based on his lectures in Cambridge. In these books he (very informally, having almost 200 explanatory figures inside 100 pages of text) explains that there is a strong connection between determining tangents and computing areas under graphs. This idea is at the core of calculus as we know it these days.



Isaac Barrow (1630–1677), courtesy of Wikipedia

Some historical notes on calculus

1684: Gottffried Wilhelm Leibniz publishes "Nova methodus pro maximis et minimis", that is "New method for maximums and minimums", which is probably the first publication about calculus.

Gottfried Wilhelm Leibniz (1646–1716), courtesy of Wikipedia

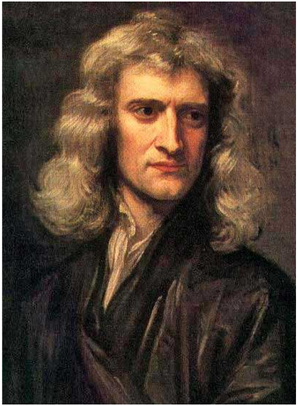
Some historical notes on calculus

1679-1680: Robert Hooke and Isaac Newton exchange a few letters discussing physics and philoshophy, in which Newton suggests Hooke to find an experimental proof of the fact that the earth revolves around its axis. Hooke responds, and formulates the gravity law as a conjecture. This inspires Newton to return from alchemy to physics, and lay out foundations of theoretical physics the way we know it.

Robert Hooke (1635–1703), courtesy of Wikipedia

Some historical notes on calculus

1687: Isaac Newton publishes “Philosophiæ Naturalis Principia Mathematica”, that is “Mathematical Principles of Natural Philosophy”. This book is one of the foundational writings for both theoretical physics and calculus (although it does not discuss calculus directly).



Isaac Newton (1642–1727), courtesy of Wikipedia

Some historical notes on calculus

1696: Guillaume de l'Hôpital publishes "Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes", that is "Infinitesimal calculus with applications to curved lines", the firt ever textbook on calculus. His contributions to what is contained there are essentially expository: the main contributor to the mathematical content of the book going beyond the work of Leibniz and Newton was Johann Bernoulli, hired by l'Hôpital to explain him contemporary maths of those times.

Johann Bernoulli (1667–1748), courtesy of Wikipedia

Basics of differential calculus

Informal definition. Given a function f, the tangent line to its graph at $x = x_0$ is the limit of lines passing through the points $(x_0, f(x_0))$ and $(x, f(x))$ as x approaches x_0.

Definition. Given a function f, the tangent line at $x = x_0$ is the line defined by the equation

$$y - f(x_0) = m_{\tan}(x - x_0),$$

where

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided this limit exists.

Alternatively, denoting $x = x_0 + h$, where h is thought as approaching 0, the formula above can be written as

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Basics of differential calculus

If $f(x)$ describes the position of a particle moving along the line after x units of time elapsed, the quantity $\frac{f(x)-f(x_0)}{x-x_0}$ is the "average velocity" during the period from x_0 to x, and the limit

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is "instantaneous velocity" at time x_0.

Generally, if $f(x)$ denotes the way a certain quantity f changes depending on a parameter x (e.g. population growth with time, measurements of a metal shape changing depending on temperature, cost change depending on quantity of product manufactured), the quantity $\frac{f(x)-f(x_0)}{x-x_0}$ has the meaning of "rate of change" of f with respect to x over the interval $[x_0, x]$, and

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

has the meaning of "instantaneous rate of change" of f at x_0.

Basics of differential calculus

It is important to realise that the limit $m_{\tan}$ we discussed does clearly depend on x_0, so is actually a new function. Let us state clearly its definition (replacing x_0 by x to emphasize the function viewpoint).

Definition. Given a function f, the function f' defined by the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is called the *derivative of f with respect to x*. The domain of f' consists of all x for which the limit exists.

Definition. A function f is said to be *differentiable at x_0* if the limit

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

exists. If f is differentiable at each point of the open interval (a,b), we say that f is differentiable on (a,b). (Here we in principle allow infinite intervals too, but not infinite limits!)

Some examples

Example 1. Let $f(x) = x^2$. Then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

Example 2. Let $f(x) = \frac{2}{x}$. Then

$$f'(x) = \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \lim_{h\to 0} \frac{\frac{2x-2(x+h)}{x(x+h)}}{h} =$$

$$= \lim_{h\to 0} \frac{-2h}{hx(x+h)} = \lim_{h\to 0} \frac{-2}{x(x+h)} = -\frac{2}{x^2}.$$

Example 3. Let $f(x) = \sqrt{x}$. Then

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\frac{(x+h)-x}{\sqrt{x+h}+\sqrt{x}}}{h} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$