1S11: Calculus for students in Science

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TCD

Lecture 15

Derivative and tangent lines

From the last class, we can easily derive the following recipe for writing the equation of a tangent line to the graph of f at $x = x_0$ when we can compute the derivative function:

- Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.
- Evaluate $f'(x_0)$; that is the slope of the tangent line.
- Use the point of tangency and the slope in the point–slope equation of the line to get

  $$y - f(x_0) = f'(x_0)(x - x_0),$$

  or equivalently

  $$y = f'(x_0)(x - x_0) + f(x_0).$$

Points of non-differentiability 1

A function is not differentiable if the limit defining the derivative function does not exist at the point x_0. Of course, that can happen merely because the values of $\frac{f(x_0+h)-f(x_0)}{h}$ do not have any meaningful behaviour at all. However, there are two frequent situations of that sort that deserve a special mention.

First, it may be the case that the limit on the left and on the right exist but are not equal. This would correspond to the fact that there are two different tangent lines on the left and on the right, forming a "corner point". A typical example of that sort is $f(x) = |x|$ at $x = 0$. Indeed,

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{x-0}{x} = \lim_{x\to 0^+} 1 = 1,$$
$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{-x-0}{x} = \lim_{x\to 0^-} -1 = -1.$$

Points of non-differentiability 2

A function is not differentiable if the limit defining the derivative function does not exist at the point x_0. Of course, that can happen merely because the values of $\frac{f(x_0+h)-f(x_0)}{h}$ do not have any meaningful behaviour at all. However, there are two frequent situations of that sort that deserve a special mention.

Second, it may be the case that the limit exists but is infinite. This would correspond to the fact that the graph has a vertical tangent line (such a line does not have a finite slope). A typical example of that sort is $f(x) = \sqrt[3]{x}$ at $x = 0$. Indeed,

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x\to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Differentiability and continuity

Theorem. A function f differentiable at $x = x_0$ is continuous at $x = x_0$.
Proof. We are given that $f'(x_0)$ is defined, so the limit $\lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists. To show that f is continuous at $x = x_0$, we need to show that $\lim\limits_{x\to x_0} f(x) = f(x_0)$, or equivalently that

$$\lim_{x\to x_0}(f(x) - f(x_0)) = 0.$$

Substituting $x = x_0 + h$, we see that the latter is equivalent to

$$\lim_{h\to 0}(f(x_0+h) - f(x_0)) = 0.$$

Now we can use arithmetics of limits:

$$\lim_{h\to 0}(f(x_0+h) - f(x_0)) = \lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}\cdot h\right] =$$

$$= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\lim_{h\to 0} h = f'(x_0)\cdot 0 = 0.$$

Differentiability and continuity

The converse of our theorem is not true: there exist continuous functions that are not differentiable.

The simplest examples are the functions $f(x) = |x|$ and $f(x) = \sqrt[3]{x}$ which are continuous everywhere but, as we saw, are not differentiable at $x = 0$.

For a long time it was believed that a function which is continuous would only have a few corner points and points of vertical tangency, and between those points the graph will consist of "smooth" pieces. However, it turns out that there exist continuous functions that are not differentiable anywhere. A graph of such a function is impossible to plot precisely, but they appear in real life applications, in the context of *fractals* (search for "fractals" on Google Images for some absolutely stunning pictures) and *Brownian motion of particles*.

Differentiability on a closed interval

As we mentioned last time, for a function defined on a closed interval, one cannot talk about derivatives at endpoints, since the values of the derivative function are two-sided limits. For that, one needs the "left-hand derivative" $f'_-(x)$ and the "right-hand derivative" $f'_+(x)$ that are defined as follows:

$$f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h},$$
$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}.$$

Geometrically, $f'_-(x)$ is the limit of slopes of secant lines as the point is approached from the left, and $f'_+(x)$ is the limit of slopes of secant lines as the point is approached from the right. A function defined on a closed interval $[a, b]$ is said to be differentiable if

- it is differentiable on the open interval,
- the one-sided derivatives $f'_+(a)$ and $f'_-(b)$ exist.

Other notation for derivatives

Since the goal of this module is to prepare you for dealing with real life applications of calculus, it is important to not only learn what they mean by derivative but also to get used to different notations for derivatives that you may encounter in textbooks, reference manuals etc.

Differentiation as operation:

$$f'(x) = \frac{d}{dx}[f(x)], \quad f'(x) = D_x[f(x)], \quad f'(x) = \partial_x[f(x)].$$

Differentiation through the dependent variable: if there is a dependent variable $y = f(x)$, one may use the notation

$$f'(x) = y'(x), \quad f'(x) = \frac{dy}{dx}.$$

In the latter case, the fraction $\frac{dy}{dx}$ should not really be viewed as the ratio of two (undefined) quantities dy and dx, but rather as a symbol for the derivative.

Other notation for derivatives

With all the notation we introduced, evaluating the derivative at the point x_0 in case of a complicated notation for differentiation is frequently denoted by $\Big|_{x=x_0}$, that is

$$f'(x_0) = \frac{d}{dx}[f(x)]\Big|_{x=x_0},\ f'(x_0) = D_x[f(x)]|_{x=x_0},\ f'(x_0) = \partial_x[f(x)]|_{x=x_0}$$

and

$$f'(x_0) = y'(x_0), \quad f'(x_0) = \frac{dy}{dx}\Big|_{x=x_0}.$$

Derivatives of scalar factors

Theorem. Suppose that the function f is differentiable at $x = x_0$. Then for any constant c the function cf is differentiable at $x = x_0$, and

$$(cf)'(x_0) = cf'(x_0).$$

Proof. We have

$$(cf)'(x_0) = \lim_{x \to x_0} \frac{(cf)(x) - (cf)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{cf(x) - cf(x_0)}{x - x_0} =$$
$$= \lim_{x \to x_0} c\frac{f(x) - f(x_0)}{x - x_0} = c \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = cf'(x_0).$$

Derivatives of sums and differences

Theorem. Suppose that two functions f and g are both differentiable at $x = x_0$. Then $f + g$ and $f - g$ are differentiable at $x = x_0$, and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0), \quad (f-g)'(x_0) = f'(x_0) - g'(x_0).$$

Proof. We have

$$\begin{aligned}
(f+g)'(x_0) &= \lim_{x\to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \\
&= \lim_{x\to x_0} \frac{f(x) + g(x) - f(x_0) - g(x_0)}{x - x_0} = \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0) + g(x) - g(x_0)}{x - x_0} = \\
&= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right) = \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0) + g'(x_0),
\end{aligned}$$

and the same works for $f - g$.

Derivatives of polynomials

Theorem. Every polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is differentiable everywhere, and

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}.$$

Proof. It is enough to prove that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$, since then we can use rules for scalar factors and addition. We note that similar to the fact that $a^2 - b^2 = (a-b)(a+b)$, we have

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$

$$\ldots$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}),$$

so

$$\lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} (x^{n-1} + x^{n-2}x_0 + \cdots + x x_0^{n-2} + x_0^{n-1}) = n x_0^{n-1}.$$