1S11: Calculus for students in Science

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TCD

Lecture 18

Derivatives of power functions

Last time we proved that if the function g is differentiable at $x = x_0$, and that $g(x_0) \neq 0$. Then $\frac{1}{g}$ is differentiable at $x = x_0$, and

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}.$$

We can apply this result to the function $g(x) = x^n$ which we already know to be differentiable; of course we should assume that $x \neq 0$ for $\frac{1}{g}$ to be defined. We get

$$\left(\frac{1}{x^n}\right)' = -\frac{nx^{n-1}}{(x^n)^2} = -\frac{nx^{n-1}}{x^{2n}} = -\frac{n}{x^{n+1}}.$$

In fact, if we write

$$\frac{1}{x^n} = x^{-n},$$

the formula we obtained becomes the same formula that we had for power functions with positive integer exponent:

$$(x^m)' = mx^{m-1}.$$

Chain rule

Theorem. Suppose that the function g is differentiable at x_0, and the function f is differentiable at $g(x_0)$. Then $f \circ g$ is differentiable at x_0, and

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

"Proof." We have

$$\begin{aligned}
\lim_{x\to x_0} \frac{(f \circ g)(x) - (f \circ g)(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} = \\
&= \lim_{x\to x_0} \left[\frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \frac{g(x) - g(x_0)}{x - x_0} \right] = \\
&= \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \\
&= \lim_{t=g(x)\to g(x_0)} \frac{f(t) - f(g(x_0))}{t - g(x_0)} g'(x_0) = f'(g(x_0))g'(x_0).
\end{aligned}$$

Alas, there is a big problem with this proof!

Chain rule: gap in the proof

For example, if $g(x) = c$ is a constant, our computation suggests to do the following:

$$\lim_{x\to x_0} \frac{(f\circ g)(x) - (f\circ g)(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} =$$
$$= \lim_{x\to x_0} \left[\frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \frac{g(x) - g(x_0)}{x - x_0}\right] = \dots$$

that is we divide and multiply by $g(x) - g(x_0) = c - c = 0$!! Multiplying both the numerator and the denominator by zero is not a valid way to deal with fractions.

The same gap would present itself whenever $g(x)$ is equal to $g(x_0)$ at infinitely many points x that are as close to x_0 as we want. Therefore, we need to improve our strategy of proof.

Chain rule: fixing the gap

Theorem. A function f is differentiable at $x = x_0$ with derivative $f'(x_0) = m$ if and only if

$$f(x) = f(x_0) + m(x - x_0) + r(x)(x - x_0),$$

where $\lim_{x \to x_0} r(x) = 0$.

Proof. The formula above can be rewritten as

$$f(x) - f(x_0) = m(x - x_0) + r(x)(x - x_0),$$

which for $x \neq x_0$ is the same as

$$\frac{f(x) - f(x_0)}{x - x_0} = m + r(x),$$

and now it is clear that $\lim_{x \to x_0} r(x) = 0$ if and only if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = m,$$

that is differentiability at x_0 with derivative equal to m.

Chain rule: fixing the gap

In general, the point-slope formula suggests that

$$y = f(x_0) + m(x - x_0)$$

is the equation of a line with slope m passing through $(x_0, f(x_0))$, so

$$f(x) = f(x_0) + m(x - x_0) + r(x)(x - x_0),$$

with

$$\lim_{x \to x_0} r(x) = 0$$

means that computing derivatives really means finding a line that would approximate the given function with an error of a very low magnitude (as small as we want relative to $x - x_0$).

Chain rule: fixing the gap

Using the result we just proved, the proof of the chain rule can be fixed: since g is differentiable at x_0 and f is differentiable at $g(x_0)$, we have

$$f(t) = f(g(x_0)) + f'(g(x_0))(t - g(x_0)) + r(t)(t - g(x_0)),$$
$$g(x) = g(x_0) + g'(x_0)(x - x_0) + s(x)(x - x_0)$$

where $\lim_{t \to g(x_0)} r(t) = 0$ and $\lim_{x \to x_0} s(x) = 0$. We shall rewrite those as

$$f(t) = f(g(x_0)) + (f'(g(x_0)) + r(t))(t - g(x_0)),$$
$$g(x) = g(x_0) + (g'(x_0) + s(x))(x - x_0),$$

obtaining

$$\begin{aligned}
f(g(x)) &= f(g(x_0)) + (f'(g(x_0)) + r(g(x)))(g(x) - g(x_0)) = \\
&= f(g(x_0)) + (f'(g(x_0)) + r(g(x)))((g'(x_0) + s(x))(x - x_0) = \\
&= f(g(x_0)) + f'(g(x_0))g'(x_0)(x - x_0) + \\
&+ (f'(g(x_0))s(x) + r(g(x))g'(x_0) + r(g(x))s(x))(x - x_0).
\end{aligned}$$

Chain rule: fixing the gap

We just obtained

$$f(g(x)) = f(g(x_0)) + f'(g(x_0))g'(x_0)(x - x_0) + \\ + (f'(g(x_0))s(x) + r(g(x))g'(x_0) + r(g(x))s(x))(x - x_0).$$

Note that if we define $r(g(x_0)) = 0$ and $s(x_0) = 0$, these newly defined functions are continuous at the respective points. Therefore

$$\lim_{x\to x_0}(f'(g(x_0))s(x) + r(g(x))g'(x_0) + r(g(x))s(x)) =$$
$$= f'(g(x_0))\lim_{x\to x_0} s(x) + g'(x_0)\lim_{x\to x_0} r(g(x)) + \lim_{x\to x_0} r(g(x))\lim_{x\to x_0} s(x) =$$
$$= 0 + g'(x_0)r\left(\lim_{x\to x_0} g(x)\right) + r\left(\lim_{x\to x_0} g(x)\right)\lim_{x\to x_0} s(x) =$$
$$= g'(x_0)r(g(x_0)) + r(g(x_0))\cdot 0 = 0,$$

and the chain rule $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$ follows.

Chain rule: some examples

Example 1. If $f(x) = \cos(x^3)$, then

$$f'(x) = -\sin(x^3) \cdot 3x^2 = -3x^2 \sin(x^3).$$

Example 2. If $f(x) = \tan^2 x$, then

$$f'(x) = \left((\tan x)^2\right)' = 2\tan x \cdot \frac{1}{\cos^2 x} = \frac{2\sin x}{\cos^3 x}.$$

Example 3. If $f(x) = \sqrt{x^2+1}$, then

$$f'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

Chain rule and other notation for derivatives

With the notation $\frac{dy}{dx}$ the chain rule is the easiest to memorise: if $y = f(g(x))$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

In this form, we shall use it later for integral calculus.

With the notation $\frac{d}{dx}[f(x)]$, one writes the chain rule as

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}.$$

In this form, it is useful for differentiating complicated functions that are functions of other functions. For example, $\frac{d}{dx}[u^k] = ku^{k-1}u'(x)$.

Derivatives of implicitly defined functions

Sometimes a function f is defined only implicitly, which makes it difficult to compute derivatives. A good example is that of inverse functions: the function $f(x) = x^3 + x$ is increasing and therefore one-to-one, therefore has an inverse, but it is hard to write it explicitly, let alone compute its derivative.

However, it is true that if a differentiable function admits an inverse, that inverse is *usually* differentiable: geometrically, differentiability means existence of a tangent line, and since computing the inverse is merely the reflection of a graph, it does not affect the existence of a tangent line.

The only potential problem is that some of the tangent lines would have the infinite slope, so the derivative function will not be defined. (That is what "usually" is referring to above).

Derivatives of implicitly defined functions

Example 1. For the function $f(x) = \sqrt[3]{x}$, the inverse of the function $g(x) = x^3$, we have

$$g(f(x)) = f(x)^3 = x,$$

so by chain rule (if we accept that the inverse is differentiable)

$$3f(x)^2 f'(x) = (f(x)^3)' = 1,$$

and therefore

$$f'(x) = \frac{1}{3f(x)^2} = \frac{1}{3\sqrt[3]{x^2}}.$$

Note that even though f is defined at $x = 0$, f' is not defined but has the infinite limit, which corresponds to a vertical tangent line.

Derivatives of power functions

Following the strategy from the previous slide, we can obtain

$$(\sqrt[n]{x})' = \frac{1}{n\sqrt[n]{x^{n-1}}},$$

and moreover by chain rule

$$(\sqrt[n]{x^m})' = \frac{1}{n\sqrt[n]{(x^m)^{n-1}}} \cdot mx^{m-1} = \frac{m}{n}\sqrt[n]{\frac{(x^{m-1})^n}{(x^m)^{n-1}}} = \frac{m}{n}\sqrt[n]{x^{m-n}}.$$

In fact, if we write

$$\sqrt[n]{x} = x^{\frac{1}{n}}, \quad \sqrt[n]{x^m} = x^{\frac{m}{n}}$$

these formulas become the same formula that we had for power functions with integer exponent:

$$(x^r)' = rx^{r-1}.$$

This formula turns out to be true for any exponent r.