

1S11: CALCULUS FOR STUDENTS IN SCIENCE

Dr. Vladimir Dotsenko

TCD

Lecture 21

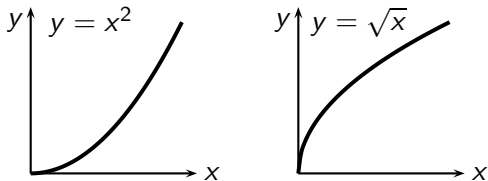
AN IMPORTANT ANNOUNCEMENT

There will be no calculus lecture on Monday November 18 at 4pm (Monday linear algebra lecture, and all lectures from Tuesday on will take place as planned).

Make sure you use that free time for revision. In particular, look at the exercises after sections 3.1 and 3.2 of the textbook. If you do not have the book, find someone who has, and make xerox copies of exercise pages (3 pages for each chapter).

CONCAVITY UP AND DOWN

The next step in analysing a function f after determining the regions where it increases and decreases, and figuring out its relative extrema is answering some finer questions, e.g. understanding in which direction the graph of f is curved. For example, the graphs of the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$ on $[0, 1]$ share many similarities, both functions being increasing, and assuming the same values at endpoints. However, the first of these graphs gets more and more steep as x increases, and the second one gets less and less steep:



Rigorously, a differentiable function is said to be *concave up* if its derivative is increasing, and *concave down* if its derivative is decreasing.

CONCAVITY UP AND DOWN

Theorem. Let f be a twice differentiable function on an open interval (a, b) .

- If $f''(x) > 0$ for each x in (a, b) , then f is concave up on (a, b) .
- If $f''(x) < 0$ for each x in (a, b) , then f is concave down on (a, b) .

This statement is true because of the connection between the property of being increasing and the sign of the derivative; we just apply the respective result to the function $g(x) = f'(x)$.

Example. For the function $f(x) = x^2 - 6x + 5$, we have $f'(x) = 2x - 6$ and $f''(x) = 2$, so this function is concave up on $(-\infty, +\infty)$.

CONCAVITY UP AND DOWN: INFLECTION POINTS

Example. For the function $f(x) = x^3$, we have $f'(x) = 3x^2$ and $f''(x) = 6x$, so this function is concave up on $(0, +\infty)$, and is concave down on $(-\infty, 0)$.

In general, points where the direction of concavity changes are of special interest; they are called *inflection points*.

In words, an inflection point marks the places on the curve $y = f(x)$ where the rate of change of y with respect to x changes from increasing to decreasing, or vice versa.

For example, if $f(x)$ is the height of the water level in a flask of a complex shape when it contains x units of water, the inflection points of the graph of $f(x)$ correspond to heights at which the flask is the most narrow, or the most wide.



INFLECTION POINTS: EXAMPLE

Example. Let us analyse the inflection points of the function $f(x) = \frac{x}{2} - \sin x$ on $[0, 2\pi]$ that we considered earlier. We have

$$f'(x) = \frac{1}{2} - \cos x,$$

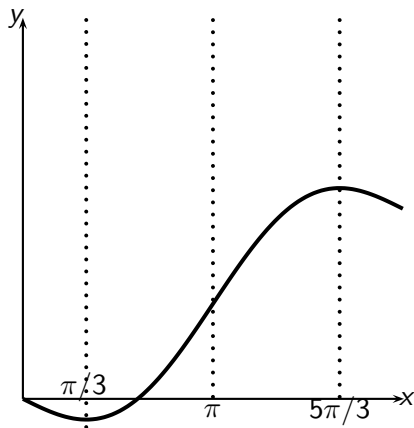
and

$$f''(x) = \sin x.$$

The second derivative vanishes at $x = \pi$, is positive for x between 0 and π , and is negative between π and 2π , so f is concave up on $[0, \pi]$, concave down on $[\pi, 2\pi]$, and has π as an inflection point.

INFLECTION POINTS: EXAMPLE

The information we gathered about $f(x) = \frac{x}{2} - \sin x$ on $[0, 2\pi]$ (increasing and decreasing, local extrema, concavity up and down, inflection points) allows to plot an approximate graph of that function:



MULTIPLICITY OF ROOTS AND GEOMETRY

A polynomial of degree n has at most n real roots, counting with multiplicities. (Recall that a real number r is said to be a root of a given polynomial p of multiplicity m if $(x - r)^m$ divides $p(x)$ but $(x - r)^{m+1}$ does not.) Our goal now is to establish how the algebraic notion of multiplicity can be interpreted geometrically.

Theorem. Suppose that r is a root of $p(x)$ of multiplicity m .

- If $m = 2k$ is even, the graph of $y = p(x)$ is tangent to the x -axis at $x = r$, does not cross the x -axis there, and does not have an inflection point there. (Think of something like $y = x^{2k}$ at $x = 0$.)
- If $m = 2k + 1 > 1$ is odd, the graph of $y = p(x)$ is tangent to the x -axis at $x = r$, crosses the x -axis there, and has an inflection point there. (Think of something like $y = x^{2k+1}$ at $x = 0$.)
- If $m = 1$ (*simple root*), then the graph of $y = p(x)$ is not tangent to the x -axis at $x = r$, crosses the x -axis there, and may or may not have an inflection point there.

ANALYSIS OF POLYNOMIALS

The information on how interpret the multiplicities of roots geometrically is possibly the most important part about graphing polynomials. Otherwise, polynomials share many “nice properties”. The natural domain of each polynomial is the whole real line, polynomials are continuous and differentiable everywhere, they increase or decrease without bound when $x \rightarrow \pm\infty$, etc. A polynomial of degree n has at most n roots (x -intercepts), at most $n - 1$ relative extrema, and at most $n - 2$ inflection points.

