1S11: Calculus for students in Science

Dr. Vladimir Dotsenko

TCD

Lecture 26

An important announcement

There will be no calculus lecture on Thursday November 28 at 9am (all other lectures and tutorials this week will take place as planned).

Make sure you use that free time (or its equivalent when you are awake. . .) for revision. In particular, look at the exercises after sections 4.3, 7.1, and 7.2 of the textbook. If you do not have the book, find someone who has, and make xerox copies of exercise pages (3 pages for each chapter).

Mnemonics for antiderivatives

Recall the notation for derivatives: if $y = f(x)$, we write $f'(x) = \frac{dy}{dx}$. Using this notation, it is possible to develop some handy mnemonic rules for evaluating indefinite integrals. We shall utilise this notation in the differential form

$$dy = f'(x)\, dx.$$

Note that the latter formula should not regarded as an equation for numbers: neither dy nor dx are numbers, they are just placeholders for easy book-keeping.

Mnemonics for antiderivatives: example

Let us see what this notation reveals about u-substitutions.

Theorem. (u-substitution) Suppose that $F(x)$ is an antiderivative of $f(x)$. Then

$$\int [f(g(x))g'(x)]\, dx = F(g(x)) + C.$$

This theorem now states that our notation $dy = f'(x)\, dx$ does not lead to contradictions. Namely, if $u = g(x)$, then the integral $\int f(u)\, du$ can be evaluated in two ways:

- we can find an antiderivative F of $f(x)$, and substitute $u = g(x)$ in it,
- or we can write $f(u)\, du = f(g(x))g'(x)\, dx$, and find an antiderivative of $f(g(x))g'(x)$.

Our theorem guarantees that the two results are the same: the two antiderivatives differ by a constant.

More examples of u-substitution

Example. Let us evaluate the integral

$$\int \frac{1}{x\ln x}\,dx.$$

Putting $t = \ln x$, so that $x = e^t$, and $dx = e^t\,dt$, we have

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{e^t\cdot t}e^t\,dt = \int \frac{dt}{t} = \ln t + C,$$

so that

$$\int \frac{1}{x\ln x}\,dx = \ln(\ln x) + C.$$

More examples of u-substitution

Example. Let us evaluate the integral

$$\int x^2\sqrt{x-1}\,dx.$$

Putting $t=\sqrt{x-1}$, so that $x=t^2+1$, and $dx=2t\,dt$, we have

$$\int x^2\sqrt{x-1}\,dx =$$

$$= \int (t^2+1)^2\cdot t\cdot 2t\,dt = \int (t^4+2t^2+1)\cdot 2t^2\,dt =$$

$$= \int (2t^6+4t^4+2t^2)\,dt = \frac{2}{7}t^7+\frac{4}{5}t^5+\frac{2}{3}t^3+C =$$

$$= \frac{2}{7}(x-1)^{7/2}+\frac{4}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+C.$$

Exercise. Compare this with Example 9 of Section 4.3 in the book, where a different u-substitution is used.

More examples of u-substitution

Example. Let us evaluate the integral

$$\int (\cos x)^3 \, dx.$$

Noticing that $\cos x \, dx = d(\sin x)$, we put $u = \sin x$ and rewrite our integral as

$$\int (\cos x)^2 \, d(\sin x) = \int (1 - (\sin x)^2) \, d(\sin x) = \int (1 - u^2) \, du.$$

The latter integral can be evaluated directly:

$$\int (1 - u^2) \, du = u - \frac{u^3}{3} + C,$$

so

$$\int (\cos x)^3 \, dx = \sin x - \frac{1}{3}(\sin x)^3 + C.$$

More examples of u-substitution

Example. Let us evaluate the integral

$$\int \sin(\ln x)\, dx.$$

Putting $t = \ln x$, so that $x = e^t$, and $\frac{dx}{dt} = e^t$, we get

$$\int \sin(\ln x)\, dx = \int \sin t\, e^t\, dt.$$

That latter integral we computed using integration by parts yesterday;

$$\int e^t \sin t\, dt = \frac{1}{2}(e^t \sin t - e^t \cos t) + C,$$

so

$$\int \sin(\ln x)\, dx = \frac{1}{2}(x \sin(\ln x) - x \cos(\ln x)) + C.$$

Mnemonics for antiderivatives, continued

Let us utilise our notation to obtain a nice way for memorising integration by parts.

Theorem. (Integration by parts) Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively. Then

$$\int [f(x)G(x)]\,dx = F(x)G(x) - \int [F(x)g(x)]\,dx,$$

This formula now becomes

$$\int G\,dF = FG - \int F\,dG.$$

More examples of indefinite integrals

Example. Let us evaluate the integral

$$\int x^3 e^{x^2}\, dx.$$

Let us use the u-substitution $u = x^2$, so that $du = 2x\, dx$. Our integral can be then rewritten as

$$\int x^2 e^{x^2} \cdot x\, dx = \int \frac{1}{2} u e^u\, du.$$

The latter integral can be computed using integration by parts:

$$\int \frac{1}{2} u e^u\, du = \int \frac{1}{2} u\, d(e^u) = \frac{1}{2} u e^u - \int \frac{1}{2} e^u\, du = \frac{1}{2} u e^u - \frac{1}{2} e^u + C,$$

so

$$\int x^3 e^{x^2}\, dx = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C.$$

More examples of indefinite integrals

Example. Let us evaluate the integral

$$\int \sin^{-1} x \, dx.$$

Denoting $\sin^{-1} x = t$, so that $x = \sin t$, and $\frac{dx}{dt} = \cos t$, we get

$$\int \sin^{-1} x \, dx = \int t \cos t \, dt.$$

This latter integral can be dealt with using integration by parts: $\cos t \, dt = d(\sin t)$, so

$$\int t \cos t \, dt = \int t \, d(\sin t) = t \sin t - \int \sin t \, dt = t \sin t + \cos t + C.$$

Recalling that $t = \sin^{-1} x$, so $\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - x^2}$, we get

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C.$$

One slightly confusing example

Example. Let us examine the integral

$$\int \frac{1}{x \ln x}\, dx$$

which we considered earlier. We shall attempt to use integration by parts with $G = \frac{1}{\ln x}$ and $dF = \frac{1}{x}\, dx$, so that $F = \ln x$. This gives us

$$\int \frac{1}{x \ln x}\, dx = \int G\, dF = FG - \int F\, dG =$$

$$= \ln x \frac{1}{\ln x} - \int \ln x \cdot \left(-\frac{1}{(\ln x)^2}\right) \cdot \frac{1}{x}\, dx = 1 + \int \frac{1}{x \ln x}\, dx.$$

What has just happened??? Everything is fine: recall that an indefinite integral is defined up to a constant, so this formula just tells us that the integral is equal to itself, and is not helping to evaluate it.