

1S11: CALCULUS FOR STUDENTS IN SCIENCE

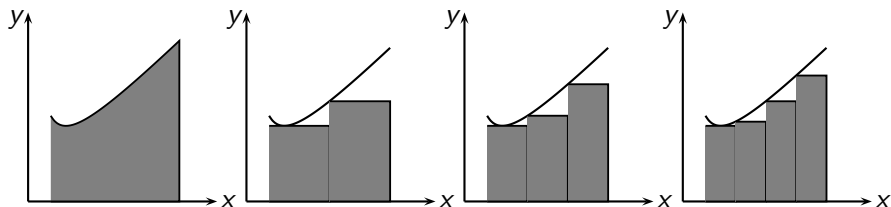
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TCD

Lecture 28

INTEGRATION

This week, we shall develop recipes for computing areas under curves. It turns out that derivatives and antiderivatives play the most important part of that story. We shall see that later on, but first let us develop a logical sequence towards computing areas.



The main idea is very simple: try to approximate curved shapes by unions of rectangular shapes that get more and more refined, and look at the limit of the areas thus obtained.

INTEGRATION: EXAMPLE

As an example, let us compute the area under $y = x^2$ when x varies from 0 to 1.

To approximate curved shapes by unions of rectangular shapes, we split the segment $[0, 1]$ into N segments of length $1/N$:

$$[0, 1/N], [1/N, 2/N], \dots, [(N-1)/N, 1].$$

On each segment $[k/N, (k+1)/N]$, let us approximate the function $f(x) = x^2$ by the value at the left end of that segment, that is $(k/N)^2$. The total approximate area is

$$\frac{1}{N} \left(0 + \frac{1}{N^2} + \frac{4}{N^2} + \dots + \frac{(N-1)^2}{N^2} \right),$$

since we look at rectangles of width $\frac{1}{N}$ and heights equal to the respective values of $f(x)$.

INTEGRATION: EXAMPLE

Theorem. We have

$$1^2 + 2^2 + 3^2 + \cdots + (N-1)^2 = \frac{(N-1)N(2N-1)}{6}.$$

Because of this theorem, the total approximate area is equal to

$$\frac{(N-1)N(2N-1)}{6N^3} = \frac{1}{6} \left(1 - \frac{1}{N}\right) \left(2 - \frac{1}{N}\right),$$

which tends to $\frac{2}{6} = \frac{1}{3}$ as N increases without bound.

We conclude that it is sensible to define the area under $y = x^2$ on $[0, 1]$ to be equal to $1/3$.

SIGMA NOTATION

As it should have become apparent by now, to deal with areas, we shall be using long sums with many terms. It is beneficial to get used to the relevant mathematical notation. Suppose that we have a function $f(x)$, and compute the sum

$$f(m) + f(m + 1) + \cdots + f(n),$$

In such a case, the symbol \sum (coming from the capital greek letter Σ (sigma) that usually denotes various kinds of sums) is used, in the following way:

$$\sum_{k=m}^n f(k) = f(m) + f(m + 1) + \cdots + f(n).$$

In these formulas, k is a “summation index”, or “summation variable”, the “ $k = m$ ” below the sigma sign indicates the starting value of k , and n above the sigma sign indicates the ending value of k .

SIGMA NOTATION

Example. For instance, the formula for the sum of squares we used earlier can be written as

$$\sum_{k=1}^{N-1} k^2 = \frac{(N-1)N(2N-1)}{6}.$$

The following theorem holds for basic sums of values of polynomials.

Theorem. The following summation formulas are true:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}.$$

Note that the second claim does not contradict our previous result: if we set $n = N - 1$, we get

$$\frac{(N-1)(N-1+1)(2(N-1)+1)}{6} = \frac{(N-1)N(2N-1)}{6}.$$

SIGMA NOTATION

Scalar factors, sums, and differences behave well with the sigma notation:

$$\sum_{k=m}^n cf(k) = c \sum_{k=m}^n f(k),$$

$$\sum_{k=m}^n (f(k) + g(k)) = \sum_{k=m}^n f(k) + \sum_{k=m}^n g(k),$$

$$\sum_{k=m}^n (f(k) - g(k)) = \sum_{k=m}^n f(k) - \sum_{k=m}^n g(k).$$

Using the basic summation formulas, and properties of sums, we can compute more sums, e.g.

$$\begin{aligned} \sum_{k=1}^n (k^2 - 2k) &= \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k = \frac{n(n+1)(2n+1)}{6} - 2 \frac{n(n+1)}{2} = \\ &= \frac{2n^3 + 3n^2 + n - 6n^2 - 6n}{6} = \frac{2n^3 - 3n^2 - 5n}{6}. \end{aligned}$$

INTEGRATION

Let us describe the general recipe for computing areas. Suppose that $f(x)$ is nonnegative on $[a, b]$, so that we can talk about the area between the graph of f and the x -axis.

We let $\Delta x = \frac{b-a}{N}$, and consider the points $x_1 = a + \Delta x$, $x_2 = x_1 + \Delta x$, \dots , $x_{N-1} = x_{N-2} + \Delta x$ that divide the interval $[a, b]$ into N subintervals of equal length Δx . Let us select arbitrary points x_1^* , \dots , x_N^* in each of the respective intervals; each such point x_k^* defines a rectangle of width Δx and height $f(x_k^*)$ which approximates the area under the graph. As an approximation to the area, we take

$$f(x_1^*)\Delta x + \dots + f(x_N^*)\Delta x = \sum_{k=1}^N f(x_k^*)\Delta x.$$

Definition. The area under the curve $y = f(x)$ on $[a, b]$ is defined as the limit

$$\lim_{N \rightarrow +\infty} \sum_{k=1}^N f(x_k^*)\Delta x.$$

Remarks:

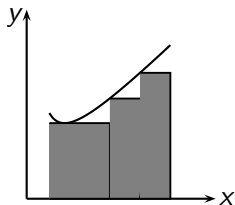
- We are using the limit $\lim_{N \rightarrow +\infty}$, where N is an integer; this is different from the limits $\lim_{x \rightarrow +\infty}$ that we considered before. Fortunately, it is possible to prove that all the properties of limits hold in this case also.
- Naturally, this limit may not exist, or may depend on the choices of points x_k^* , and so on. However, it turns out that for a continuous nonnegative function, this limit always exists, and is well defined, that is does not depend on any choices.
- If we do not assume that f is nonnegative, the sums we compute do not represent areas any longer, but rather the difference between the area above the curve and the difference below the curve. This quantity is usually referred to as the *net signed area* between the curve $y = f(x)$ and the x -axis.

INTEGRATION: RIEMANN SUMS

It turns out that for some purposes having all the integrals of the same length Δx is not optimal. To remedy this, we shall consider arbitrary points x_1, x_2, \dots, x_{N-1} that divide the interval $[a, b]$ into N subintervals of lengths

$$\Delta x_1 = x_1 - a, \Delta x_2 = x_2 - x_1, \dots, \Delta x_n = b - x_{n-1}.$$

Of course, if it is not the case, we need to somehow make sure that the sums we compute do approximate the area under the curve. A relatively neat way to do that is to require that the *mesh size* of this partition into subintervals, that is $\max \Delta x_k$, gets as close to zero as we want.



INTEGRATION: RIEMANN SUMS

The respective definition needs to be altered to use *Riemann sums*

$$\sum_{k=1}^N f(x_k^*) \Delta x_k$$

instead of $\sum_{k=1}^N f(x_k^*) \Delta x$.

Definition. A function f is said to be (*Riemann*) *integrable* if the limit

$$\lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^N f(x_k^*) \Delta x_k$$

exists, and does not depend on either the choice of partitions or the choice of points x_k^* in the respective subintervals. In that case, that limit is denoted by

$$\int_a^b f(x) dx,$$

and is referred to as the *definite integral* of $f(x)$ on $[a, b]$.

INTEGRABILITY

It is not easy to determine precisely which functions are integrable. However, the good news is that all “reasonably nice” functions are integrable:

Theorem. A function f that is continuous on $[a, b]$ is integrable on $[a, b]$. Moreover, if the function f is piecewise defined, and continuous on each piece, then it is integrable. In each of these cases, the definite integral

$$\int_a^b f(x) dx$$

is equal to the net signed area between the curve $y = f(x)$ and the x -axis.

PROPERTIES OF DEFINITE INTEGRALS

Integrals behave well with scalar factors, sums, and differences:

Theorem. Suppose that both functions f and g are integrable on $[a, b]$. Let c be a constant. Then the functions cf , $f + g$, and $f - g$ are integrable, and

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx,$$
$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx,$$
$$\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx.$$

More generally, we have

$$\int_a^b (c_1 f_1(x) + \cdots + c_n f_n(x)) dx = c_1 \int_a^b f_1(x) dx + \cdots + c_n \int_a^b f_n(x) dx.$$

PROPERTIES OF DEFINITE INTEGRALS

To simplify various formulas, it is beneficial to define the integral symbol \int_a^b beyond the case $a < b$ we have been considering now, as follows:

Definition. If a is in the domain of f , we define $\int_a^a f(x) dx = 0$. Also, if f is integrable on $[a, b]$, we define

$$\int_b^a f(x) dx = - \int_a^b f(x) dx.$$

The former formula is consistent with the intuition of areas: on the interval of zero length, the net signed area must be equal to zero. The latter formula is mostly conventional, although in some cases it has physical meaning: if $f(x)$ is the gravity force between the unit mass placed at the point x and the origin, then $\int_a^b f(x) dx$ is the work required to move the mass from a to b . Of course, moving it in the opposite direction would require negative work on our side, since it will all be done by the gravity!