1S11: Calculus for students in Science

Dr. Vladimir Dotsenko

TCD

Lecture 29

Further properties of definite integrals

Let us summarise a few further properties of definite integrals.

Theorem. Suppose that f is integrable on a closed interval containing a, b, c. Then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx,$$

no matter how the points a, b, c are ordered. (Recall the convention of changing the sign when swapping the limits of integration.)

This result has a very clear meaning from the area point of view, since for $a < b < c$ and a non-negative function f this just means that the total area under the graph on $[a, c]$ is the sum of the area on $[a, b]$ and the area on $[b, c]$.

FURTHER PROPERTIES OF DEFINITE INTEGRALS

Theorem. If f is integrable on $[a, b]$, and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x) \, dx \geq 0.$$

Moreover, if f and g are integrable on $[a, b]$, and $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx.$$

This result also has a clear interpretation in terms of areas: for a nonnegative function, the net signed area between the graph and the x-axis is nonnegative.

The fundamental theorem of calculus

The fundamental theorem of calculus relates computing derivatives, computing antiderivatives, and computing areas. It consists of two parts, that we shall discuss separately.

Theorem. (The Fundamental Theorem of Calculus, part 1) If f is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Let us outline the proof. The key ingredient is the so called Mean-Value Theorem, or Lagrange's Theorem, for derivatives, saying that if g is differentiable on $[c, d]$, then there exists a point u in $[c, d]$ for which

$$g(d) - g(c) = g'(u)(d - c),$$

in other words, that there exists a point at which the tangent line is parallel to the line connecting the endpoints of the curve $y = f(x)$ on $[a, b]$.

THE FUNDAMENTAL THEOREM OF CALCULUS

Let us subdivide $[a, b]$ into small intervals by arbitrary points

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

Applying the Mean-Value Theorem to each of the intervals and the function F, we get

$$
\begin{aligned}
F(x_1) - F(a) &= F'(x_1^*)(x_1 - a) = F'(x_1^*)\Delta x_1, \\
F(x_2) - F(x_1) &= F'(x_2^*)(x_2 - x_1) = F'(x_2^*)\Delta x_2, \\
F(x_3) - F(x_2) &= F'(x_3^*)(x_3 - x_2) = F'(x_3^*)\Delta x_3, \\
&\cdots \\
F(b) - F(x_{n-1}) &= F'(x_n^*)(b - x_{n-1}) = F'(x_n^*)\Delta x_n,
\end{aligned}
$$

so that adding these equations yields

$$F(b) - F(a) = \sum_{k=1}^{n} F'(x_k^*)\Delta x_k = \sum_{k=1}^{n} f(x_k^*)\Delta x_k,$$

since F is an antiderivative of f.

THE FUNDAMENTAL THEOREM OF CALCULUS

Finally, we note that as $\max \Delta x_k \to 0$, the right hand side of the equation

$$F(b) - F(a) = \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

has $\int_a^b f(x)\, dx$ as its limit, and the left hand side does not change, which proves the theorem.

There are several conventions to denote $F(b) - F(a)$ in a more compact way:

$$F(x)]_a^b, \quad [F(x)]_a^b, \quad F(x)|_a^b,$$

and so on.

Example. We know that $F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x) = x^2$. Therefore,

$$\int_0^1 f(x)\, dx = \frac{x^3}{3} \bigg]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3},$$

confirming our computation yesterday.

THE FUNDAMENTAL THEOREM OF CALCULUS

Note that the expression $F(b) - F(a)$ does not depend on the choice of the antiderivative (where we can add an arbitrary constant).

This allows to relate the definite and the indefinite integral (which is not a function but a family of functions!) as follows:

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Bigg]_a^b.$$

Warning. It is very important to only apply the fundamental theorem of calculus where it is applicable: e.g. if $f(x) = \frac{1}{x^2}$, so that $F(x) = -\frac{1}{x}$, then applying our theorem mechanically on $[-1, 1]$ would tell us

$$\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x} \Bigg]_{-1}^1 = -1 - 1 = -2,$$

which is clearly absurd. The problem is that our function is not even defined everywhere on $[-1, 1]$, and is certainly not continuous at all points.

THE FUNDAMENTAL THEOREM OF CALCULUS

Example. Applying the theorem we just proved, we can evaluate the following integrals:

$$\int_{-2}^{1} (x^2 - 6x + 12)\, dx = \frac{x^3}{3} - 3x^2 + 12x \Bigg]_{-2}^{1} =$$

$$= \frac{1}{3} - 3 + 12 - \left(-\frac{8}{3} - 12 - 24 \right) = 48.$$

$$\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \sin\theta]_{-\pi/2}^{\pi/2} = 1 - (-1) = 2.$$

$$\int_{1}^{4} \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}]_{1}^{4} = 4 - 2 = 2.$$

THE FUNDAMENTAL THEOREM OF CALCULUS

Example. Applying the theorem we just proved, we can evaluate the following integrals:

$$\int\limits_0^{\pi/4} \left(3x^2 + \frac{1}{\cos^2 x}\right)\, dx = x^3 + \tan x\Big]_0^{\pi/4} = \frac{\pi^3}{64} + 1.$$

$$\int\limits_0^1 |2x - 1|\, dx = \int\limits_0^{1/2} |2x - 1|\, dx + \int\limits_{1/2}^1 |2x - 1|\, dx =$$

$$= \int\limits_0^{1/2} (1 - 2x)\, dx + \int\limits_{1/2}^1 (2x - 1)\, dx = x - x^2\Big]_0^{1/2} + x^2 - x\Big]_{1/2}^1 =$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) - (0 - 0) + (1 - 1) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{2}.$$

THE FUNDAMENTAL THEOREM OF CALCULUS

There are many applications in which the fundamental theorem of calculus has a clear meaning for real life questions, all following the following general principle:

Integrating the rate of change of $F(x)$ over an interval $[a, b]$ produces the total change of the value of $F(x)$ as x increases from a to b.

- If $s(t)$ is the position of the particle in rectilinear motion, then $s'(t) = v(t)$ is the instantaneous velocity, and integrating $v(t)$ computes the displacement between the given times.
- If $P(t)$ is the population (of plants, animals, people) at time t, then $P'(t) = r(t)$ is the rate of change of population, and its integral is the total change of population.
- If $p(x)$ is the profit generated by producing and selling x units of product, then $p'(x) = m(x)$ is the "marginal profit" (extra profit resulting from producing and selling one extra unit), and its integral from a to b is the change of profit when the production level increases from a to b units.

The fundamental theorem of calculus

The remaining part of the fundamental theorem of calculus says that if we take a definite integral with varying upper limit, and differentiate it with respect to that limit, we get back the function we integrate:

Theorem. (The Fundamental Theorem of Calculus, part 2) If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point on the interval, then the function F defined by

$$F(x) = \int_a^x f(t)\, dt$$

is an antiderivative of f, that is $F'(x) = f(x)$ on $[a, b]$. In the operator notation,

$$\frac{d}{dx}\left[\int_a^x f(t)\, dt \right] = f(x).$$

THE FUNDAMENTAL THEOREM OF CALCULUS

By definition of the derivative, we have

$$
\begin{aligned}
F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} &= \\
&= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \right] = \\
&= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)\, dt + \int_x^a f(t)\, dt \right] = \\
&= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt.
\end{aligned}
$$

Now we shall use the so called Mean-Value Theorem for integrals, saying that if g is continuous on $[c, d]$, then there exists a point u in $[c, d]$ for which

$$
\int_c^d g(x)\, dx = g(u)(d - c).
$$

THE FUNDAMENTAL THEOREM OF CALCULUS

By Mean-Value Theorem, we see that

$$\int_x^{x+h} f(t)\,dt = f(u) \cdot h,$$

where u is a point from the interval $[x, x+h]$. Therefore,

$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} \frac{1}{h}(f(u) \cdot h) = \lim_{h \to 0} f(u) = f(x),$$

since u is between x and $x+h$.

The fundamental theorem of calculus

Summing up, we now fully proved that differentiation and integration are inverse procedures, having shown that

$$\int_a^b F'(x)\,dx = F(b) - F(a),$$

so that functions can be reconstructed from their derivatives by integration, and that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x),$$

so that functions can be reconstructed from their integrals by differentiation. These results, forming together "The Fundamental Theorem of Calculus", are among the most important discoveries in the history of science.