1S11: Calculus for students in Science

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TCD

Lecture 4

Functions with extra symmetries

Definition. A function f is said to be *even* if $f(-x) = f(x)$ for all x in the domain of f.

Definition. A function f is said to be *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

Definition. A function f is said to be *periodic with period T* if $f(x + T) = f(x)$ for all x in the domain of f.

More precisely, one should request that for all x in the domain of f, the value $f(-x)$ (respectively, $f(x + T)$) is defined, and is related to the value $f(x)$ as above.

Functions with extra symmetries

In terms of graphs:

- the graph of an even function is symmetric about the y-axis;
- the graph of an odd function is symmetric about the origin $(0, 0)$;
- the graph of a periodic function does not change under the horisontal shift through T units.

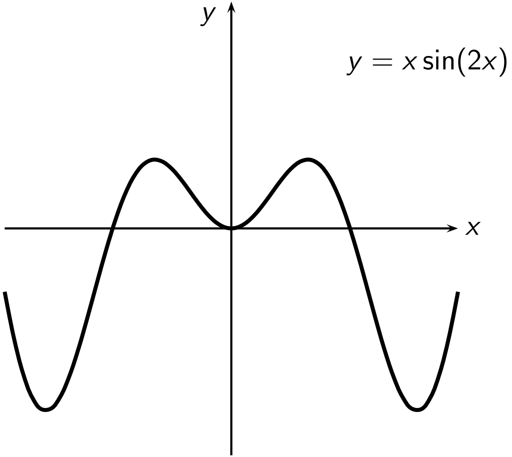
Being aware of extra symmetries allows us to study a function just on a part of its domain, and derive information elsewhere by using symmetry.

Functions with extra symmetries

Let us take the function $f(x) = x\sin(2x)$. Since

$$f(-x) = (-x)\sin(-2x) = (-x)(-\sin(2x)) = x\sin(2x),$$

this function is even. The corresponding graph looks as follows:

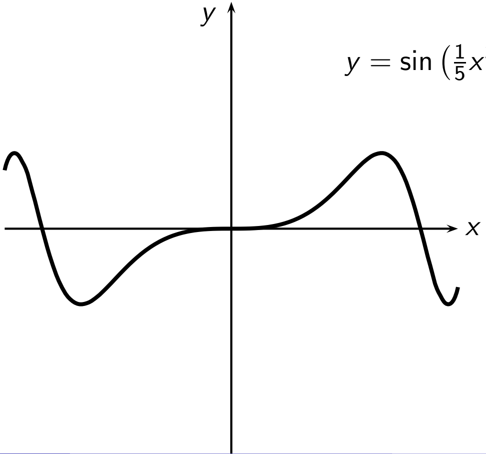


Functions with extra symmetries

Let us take the function $f(x) = \sin\left(\frac{1}{5}x^3\right)$. Since

$$f(-x) = \sin\left(\frac{1}{5}(-x)^3\right) = \sin\left(-\frac{1}{5}x^3\right) = -\sin\left(\frac{1}{5}x^3\right),$$

this function is odd. The corresponding graph looks as follows:

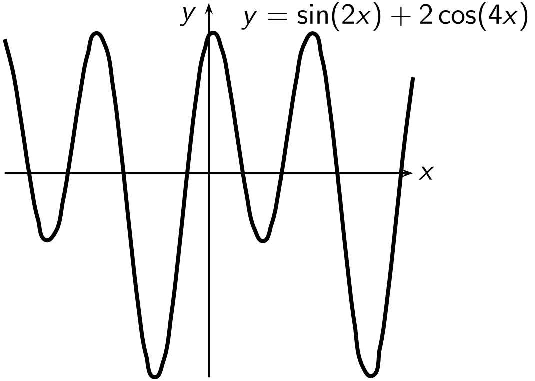


Functions with extra symmetries

Let us take the function $f(x) = \sin(2x) + 2\cos(4x)$. Since

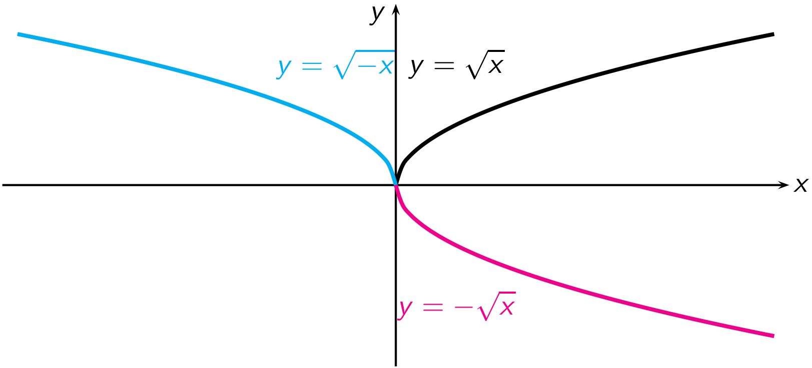
$$f(x+\pi) = \sin(2(x+\pi)) + 2\cos(4(x+\pi)) =$$
$$= \sin(2x+2\pi) + 2\cos(4x+4\pi) = \sin(2x) + 2\cos(4x) = f(x),$$

this function is periodic with period π.



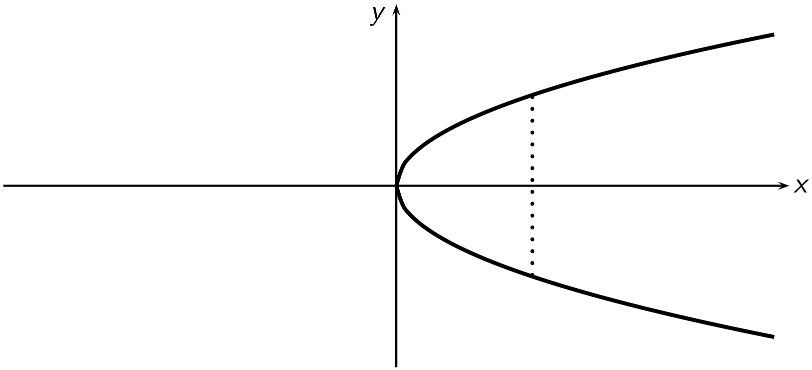
Functions with extra symmetries

What about the symmetry about the x-axis? We discussed that type of symmetry the last time:



Functions with extra symmetries

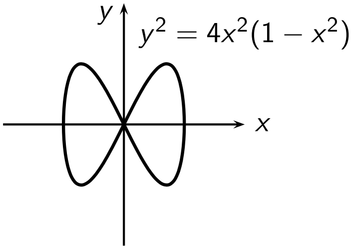
A graph of a function cannot be symmetric about the x-axis, since that would break the vertical line test:

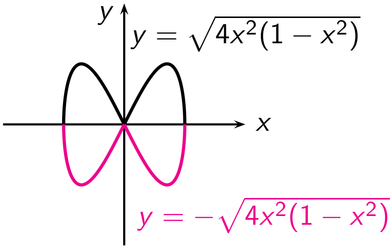


The only exception is the graph of the zero function $f(x) = 0$, since in this case each vertical line meets the graph at the x-axis, so there is just one intersection point.

Functions with extra symmetries

It can however be useful to apply the symmetry about the x-axis to plot curves defined by equations:



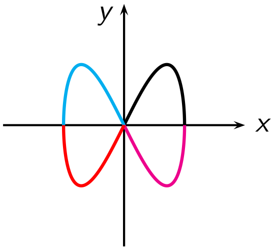


Functions with extra symmetries

For curves defined by equations we have the following symmetry tests:

- A plane curve is symmetric about the y-axis if and only if its equation does not change under replacing x by $-x$;
- A plane curve is symmetric about the x-axis if and only if its equation does not change under replacing y by $-y$;
- A plane curve is symmetric about the origin if and only if its equation does not change under replacing x by $-x$ and y by $-y$ simultaneously.

The curve $y^2 = 4x^2(1 - x^2)$ satisfies all these conditions, so it has many different symmetries:



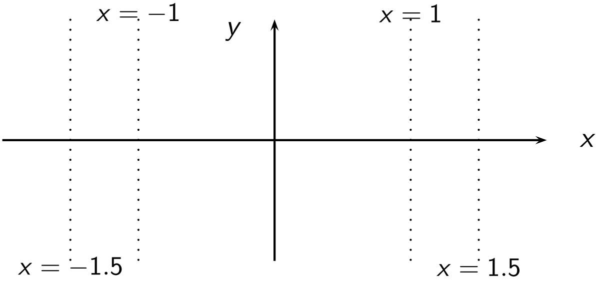
Classical parametric families of functions

Let us, alongside with building a "vocabulary" for talking about functions, start building a "library" of functions.

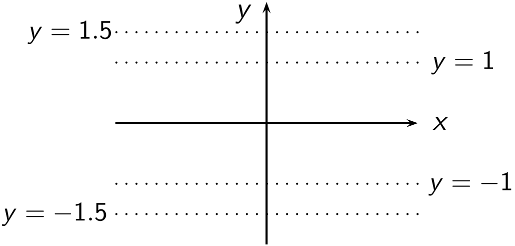
Geometrically, the simplest possible shape is the straight line.

- horisontal straight lines are defined by equations $y = c$ for various c;
- vertical lines are defined by equations $x = c$ for various c;
- general non-vertical lines are defined by equations $y = mx + b$ for various m and b;
- general lines are defined by linear equations $ax + by = c$ for various a, b, c.

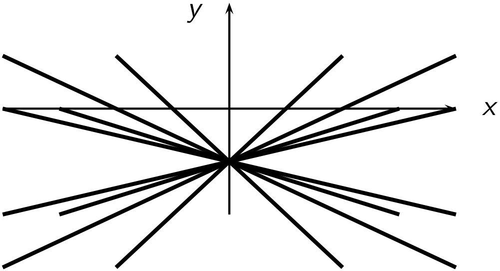
Classical parametric families of functions



Classical parametric families of functions

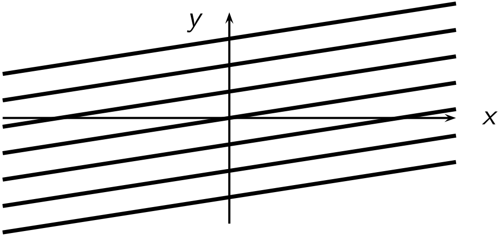


Classical parametric families of functions



These "whiskers" are lines $y = mx - 1$ for various m. Note that for $x = 0$ the formula produces -1 regardless of what m is, which agrees with the picture well.

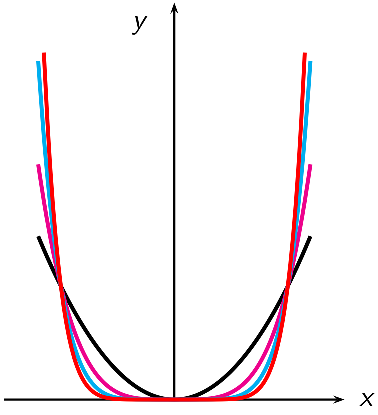
Classical parametric families of functions



These lines are $y = x/3 + b$ for various b. Note that they all are indeed obtained from $y = x/3$ by vertical shifts.

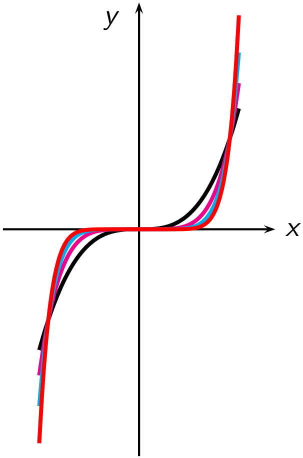
Classical parametric families of functions

Algebraically, one of the simplest functions is the power function, $y = x^n$. We shall consider several different options for n, assuming that it is an integer.



This illustrates the behaviour of $y = x^n$ for even $n = 2, 4, 6, 8$.

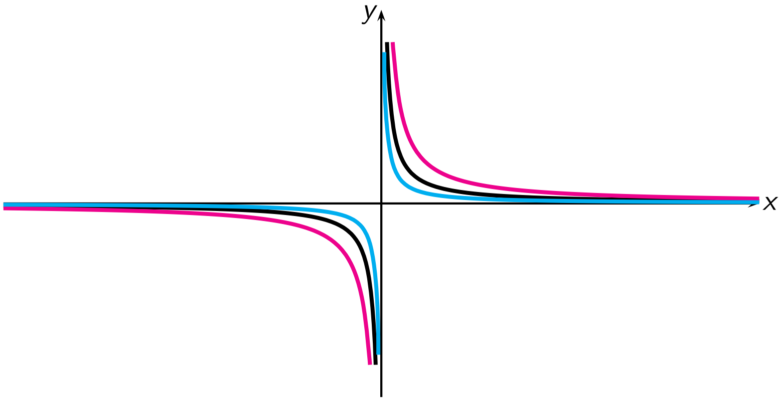
Classical parametric families of functions



This illustrates the behaviour of $y = x^n$ for odd $n = 3, 5, 7, 9$.

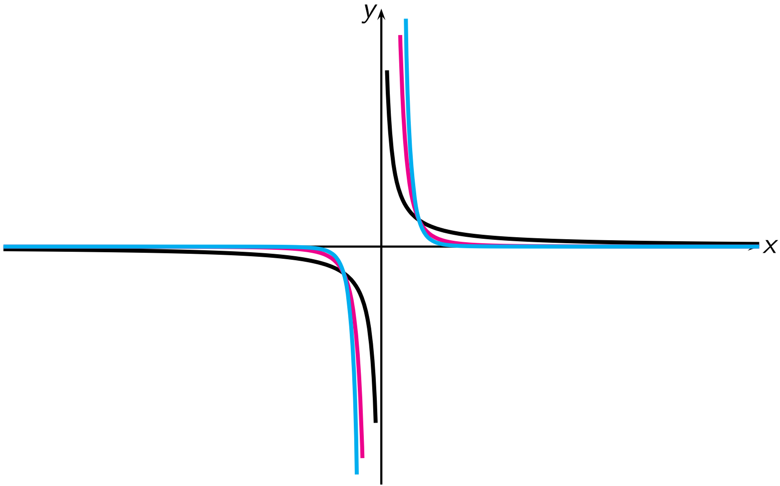
Classical parametric families of functions

The inverse proportionality $y = k/x$ is a function that appears in many situations, e.g. Boyle's Law $PV = k$ (for a fixed amount of an ideal gas at constant temperature)



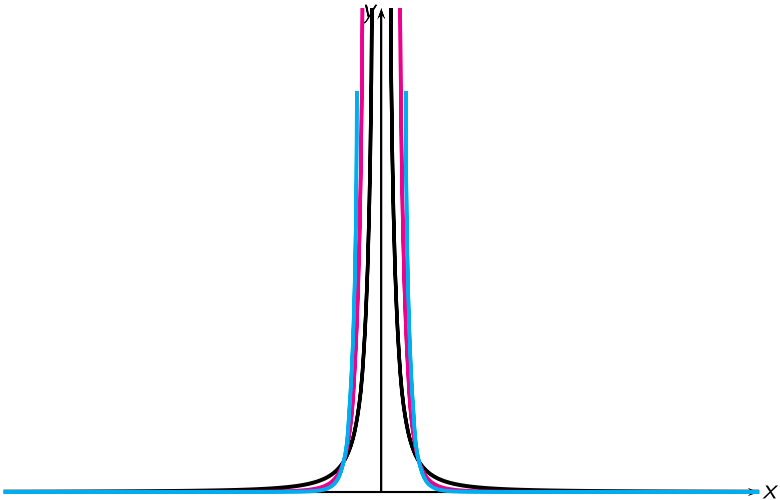
This illustrates the behaviour of $y = k/x$ for $k = 1/2, 1, 2$, the larger k, the further the graph would be from the origin.

Classical parametric families of functions



This illustrates the behaviour of $y = x^n$ for odd $n = -1, -3, -5$.

Classical parametric families of functions



This illustrates the behaviour of $y = x^n$ for even $n = -2, -4, -6$.

Classical parametric families of functions

Summary for power functions:

- for even $n \geq 2$ we get something that looks like the parabola (but is flatter close to the origin, and steeper far from the origin);
- for odd $n \geq 3$ we get something that looks like the parabola for $x > 0$ (but is flatter close to the origin, and steeper far from the origin), and is obtained by a reflection about the x-axis for $x < 0$;
- for odd $n \leq -1$, we get something that looks like the graph of inverse proportionality $y = k/x$ (but steeper close to the origin, and flatter far from the origin)
- for even $n \leq -2$, we get something that looks like the graph of inverse proportionality $y = k/x$ for $x > 0$ (but steeper close to the origin, and flatter far from the origin), and is obtained by a reflection about the x-axis for $x < 0$;
- for non-integer exponents, graphs are similar to those for integer ones; we shall discuss that in more detail a bit later.