

1S11: CALCULUS FOR STUDENTS IN SCIENCE

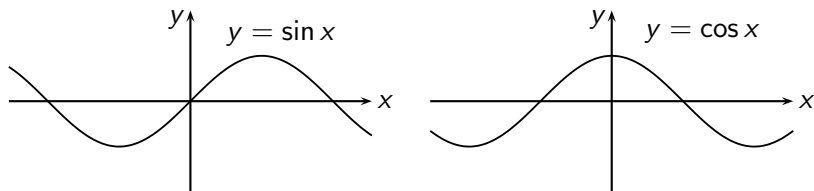
Dr. Vladimir Dotsenko

TCD

Lecture 6

CLASSICAL TYPES OF FUNCTIONS

To conclude the discussion of classical types of functions, let us discuss simplest trigonometric functions, that is \sin and \cos , and functions obtained from them by stretching/compressing and translating graphs. As we mentioned before, \sin and \cos are periodic with period 2π .



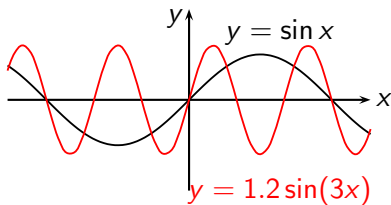
Also, \sin is an odd function, and \cos is an even function. By the way, negative angles do make perfect sense: we often need to distinguish between the clockwise direction and the counterclockwise direction. A common convention is that the clockwise direction is *negative*, and the counterclockwise direction is *positive*.

CLASSICAL TYPES OF FUNCTIONS

If we look at the function $g(x) = A \sin(Bx)$, we observe that it is $\frac{2\pi}{B}$ -periodic:

$$A \sin\left(B\left(x + \frac{2\pi}{B}\right)\right) = A \sin(Bx + 2\pi) = A \sin(Bx).$$

Also, while \sin oscillates between -1 and 1 , this new function oscillates between $-A$ and A .



The same applies to the function $A \cos(Bx)$, and also to the functions $A \sin\left(Bx - C\right) = A \sin\left(B\left(x - \frac{C}{B}\right)\right)$ and $A \cos\left(Bx - C\right) = A \cos\left(B\left(x - \frac{C}{B}\right)\right)$.

CLASSICAL TYPES OF FUNCTIONS

Besides the *period* $T = \frac{2\pi}{B}$ and the *amplitude* A , a quantity used in applications of trigonometric functions is the *frequency* $\frac{1}{T} = \frac{B}{2\pi}$, number of periods passed in one unit of time.

If A or B is negative, we should use absolute values in these formulas and write $\frac{2\pi}{|B|}$ for the period, $|A|$ for the amplitude, and $\frac{|B|}{2\pi}$ for the frequency.

INVERSE FUNCTIONS

To undo the effect of addition, we subtract: if we replace f by $f + g$, then to get back f , we subtract g :

$$(f + g) - g = f.$$

To undo the effect of multiplication, we divide: if we replace f by fg , then to get back f , we divide by g :

$$(fg)/g = f.$$

In this case, it is not always possible to undo the effect: if $g = 0$, then we cannot recover f by division.

We shall now discuss how to undo the effect of composition (when possible).

INVERSE FUNCTIONS

Let us consider an example of two different temperature scales, those of Celsius (1742) and of Fahrenheit (1724). Celsius scale, roughly speaking, sets the freezing point of water to 0° and the boiling point of water to 100° , while Fahrenheit scale sets the freezing point of water to 32° and the boiling point of water to 212° . (The zero degrees Fahrenheit refers to the freezing point of brine, the lowest temperature Fahrenheit could reliably reproduce.)

If f is the value of temperature on the Fahrenheit scale, and c is its value on the Celsius scale, then

$$f = \frac{9}{5}c + 32.$$

INVERSE FUNCTIONS

Suppose that we know the temperature on the Fahrenheit scale, and want to convert it to Celsius (someone from the US told us the temperature at their home, and we want to figure out what on earth they mean saying that it's 80 degrees... assuming they do not live in hell!)

We view the conversion formula from the previous slide as an equation for c , and solve it:

$$f = \frac{9}{5}c + 32,$$

$$f - 32 = \frac{9}{5}c,$$

$$c = \frac{5}{9}(f - 32).$$

In this case, we knew f as a function of c , and we have been able to reverse the procedure, and compute c as a function of f .

INVERSE FUNCTIONS

If we had a different functional dependence, for example,

$$f = \frac{9}{5}c^2 + 32,$$

then reconstruction of c as a function of f would not be possible, since we would get $c^2 = \frac{5}{9}(f - 32)$, which yields two possible values for c .

Definition. Suppose that for a function f there exists a function g such that

$$\begin{aligned}f(g(x)) &= x \text{ for all } x \text{ in the domain of } g, \\g(f(x)) &= x \text{ for all } x \text{ in the domain of } f.\end{aligned}$$

Then g is said to be the inverse of f , f is said to be the inverse of g , and it is always said that f and g are inverse functions.

INVERSE FUNCTIONS

Example. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then $f(g(x)) = (\sqrt{x})^2 = x$ for all x in the domain of g (that is, $x \geq 0$). However,

$$g(f(x)) = \sqrt{x^2} = |x|$$

which is different from x for $x < 0$ (which still is in the domain of f). Therefore f and g are not inverse of each other.

Example. Let $f(x) = x^2$ defined for $x \geq 0$ and $g(x) = \sqrt{x}$. Then $f(g(x)) = (\sqrt{x})^2 = x$ for all x in the domain of g (that is, $x \geq 0$). Also, $g(f(x)) = \sqrt{x^2} = x$ for $x \geq 0$ (which is the domain of f). Therefore f and g are inverse of each other, so adjusting the domain may change the property of being inverse.

INVERSE FUNCTIONS

Example. In Lecture 2, we wanted to compute the range of $f(x) = \frac{x+1}{x-1}$. For that, we solved the equation $\frac{x+1}{x-1} = a$ for a fixed value $y = a$. The answer was $x = \frac{a+1}{a-1}$. In other words, the way $y = \frac{x+1}{x-1}$ depends on x is the same as the way x depends on y . In particular, this function f is the inverse of itself:

$$f(f(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{\frac{x+1+x-1}{x-1}}{\frac{x+1-(x-1)}{x-1}} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = \frac{2x}{2} = x$$

for all $x \neq 1$.

INVERSE FUNCTIONS

It is possible to prove that if f has an inverse, then that inverse is unique. Let us sketch a proof: suppose g_1 and g_2 are two different inverses, then

$$\begin{aligned}g_1(f(g_2(x))) &= g_1(x) \text{ because } f(g_2(x)) = x \\g_1(f(g_2(x))) &= g_2(x) \text{ because } g_1(f(x)) = x.\end{aligned}$$

Therefore, the inverse of f is unique. We shall denote it by f^{-1} .

Warning: whenever f is the name of a function, f^{-1} will denote the inverse, and -1 will never mean the exponent, that is $f^{-1}(x) \neq \frac{1}{f(x)}$. This is a rather common mistake, try to refrain from it!

INVERSE FUNCTIONS

Using the notation f^{-1} for the inverse, the definition can be rewritten as

$$f(f^{-1}(x)) = x \text{ for all } x \text{ in the domain of } f^{-1},$$

$$f^{-1}(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

Example. For $c \neq 0$, the function $f(x) = cx$ has an inverse $f^{-1}(x) = \frac{1}{c}x$. Indeed,

$$f(f^{-1}(x)) = f\left(\frac{1}{c}x\right) = c \cdot \frac{1}{c}x = x,$$

$$f^{-1}(f(x)) = f^{-1}(cx) = \frac{1}{c} \cdot cx = x.$$

Example. The function $f(x) = x^3$ has an inverse $f^{-1}(x) = \sqrt[3]{x}$. Indeed,

$$f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x,$$

$$f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x.$$

INVERSE FUNCTIONS

Theorem. Suppose that a function f has an inverse f^{-1} . Then the domain of f is equal to the range of f^{-1} , and the range of f is equal to the domain of f^{-1} .

Proof. To show that the domain of f is equal to the range of f^{-1} , it is sufficient to show that every x from the domain of f is in the range of f^{-1} , and every x from the range of f^{-1} is in the domain of f .

Suppose that x is in the domain of f . Then $f^{-1}(f(x)) = x$, so x is the value of f^{-1} at the point $f(x)$, and therefore x is in the range of f^{-1} .

Suppose that x is in the range of f^{-1} , so that $x = f^{-1}(t)$ for some t . Replacing x by t in $f(f^{-1}(x)) = x$, we see that $f(f^{-1}(t)) = t$, which under our notation means $f(x) = t$, so f is defined at the point x , and therefore x is in the domain of f .

The proof of the other half of the theorem, namely that the range of f is equal to the domain of f^{-1} , is exactly the same with f and f^{-1} interchanged. □