1S11: Calculus for students in Science

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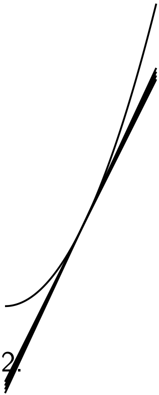
TCD

Lecture 8

Example from differential calculus: reminder

The tangent line to $y = x^2$ at the point $(1, 1)$ can be obtained as the limiting positions of the secant lines passing through the points $(1, 1)$ and $(1 + h, (1 + h)^2)$ as h gets smaller and smaller:



Here x ranges between 0 and 2.

Example from differential calculus: reminder

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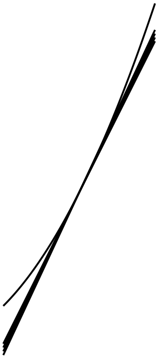
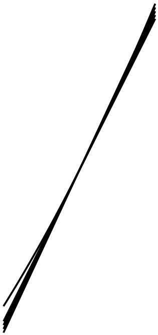


Here x ranges between 0.5 and 1.5.

Example from differential calculus: reminder

The tangent line to $y = x^2$ at the point $(1, 1)$ can be obtained as the limiting positions of the secant lines passing through the points $(1, 1)$ and $(1 + h, (1 + h)^2)$ as h gets smaller and smaller:



Here x ranges between 0.8 and 1.2.

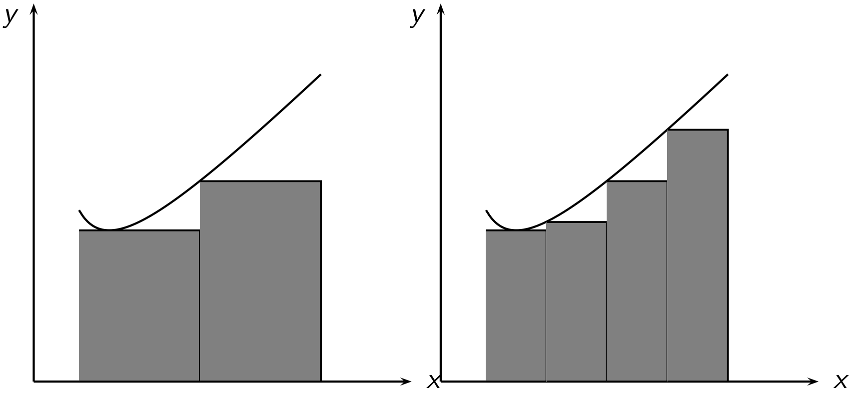
Limiting behaviour

These figures confirm our computation from yesterday: if x is close to 1, the secant lines get closer and closer to the line $y = 2x - 1$, which itself is closer and closer to the parabola.

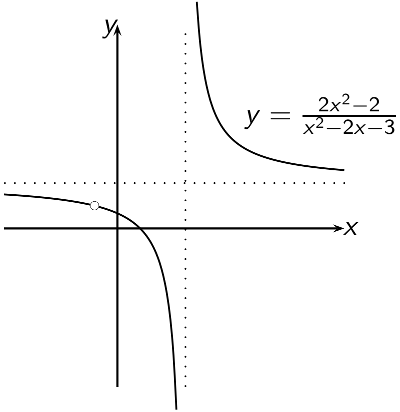
Our next goal is to make rigorous sense of these words "closer and closer", examining limiting behaviour of various functions.

Limiting behaviour

Another instance of limiting behaviour of various quantities we encounter when computing areas:



Example from Lecture 5



When x approaches -1, the value of the function

$$\frac{2x^2-2}{x^2-2x-3} = \frac{2(x-1)(x+1)}{(x+1)(x-3)} = \frac{2x-2}{x-3} = 2 + \frac{4}{x-3}$$

gets very close to 1, even though f is not defined for $x = -1$.

Limits informally

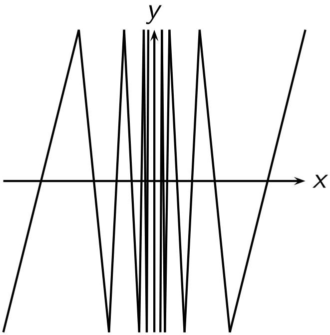
Informally, a function f is said to *have the limit L as x approaches a* if the values $f(x)$ for x sufficiently close to a are *all* as close as we like to L. In this case, one writes

$$\lim_{x \to a} f(x) = L,$$

or

$$f(x) \to L \quad \text{as} \quad x \to a.$$

The word "all" here is very important:



Limits

Example: Let us examine the limit $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$.

Values for $x < 1$ and close to 1:

x	0.99	0.999	0.9999	0.99999
$f(x)$	1.994987	1.999500	1.999950	1.999995

Values for $x > 1$ and close to 1:

x	1.00001	1.0001	1.001	1.01
$f(x)$	2.000005	2.000050	2.000500	2.004988

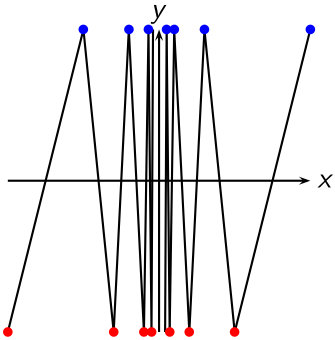
From this data, it is natural to guess that $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = 2$. Indeed, since $x - 1 = (\sqrt{x})^2 - 1$, we see that

$$\frac{x-1}{\sqrt{x}-1} = \frac{(\sqrt{x})^2-1}{\sqrt{x}-1} = \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \sqrt{x}+1$$

for $x \neq 1$, and $\sqrt{x}+1$ is very close to $\sqrt{1}+1 = 2$ for x close to 1.

Limits

Once again, it is very important that the same qualitative behaviour is observed for *all* x close to a; in the figure we saw, depending on sampling values of x we may get only values close to 1 as x approaches 0, or only values close to -1:



Limits informally and formally

Informally, a function f is said to *have the limit L as x approaches a* if the values $f(x)$ for x sufficiently close to a are *all* as close as we like to L.

More precisely: For each positive number ϵ, we can find a positive number δ so that whenever x is in $(a-\delta, a+\delta)$ (possibly with the exception of $x=a$), the value $f(x)$ belongs to $(L-\epsilon, L+\epsilon)$.

Let us note that x belongs to $(a-\delta, a+\delta)$ if and only if $|x-a|<\delta$: indeed, $|x-a|<\delta$ means $-\delta<x-a<\delta$, so $a-\delta<x<a+\delta$.

Formally: suppose that f is defined in some open interval containing the number a, except possibly for $x=a$. We shall say that f has the limit L at a, and write $\lim\limits_{x\to a} f(x)=L$, if given $\epsilon>0$, we can find $\delta>0$ such that

$$|f(x)-L|<\epsilon \text{ whenever } 0<|x-a|<\delta.$$

Similarly to the above, $|f(x)-L|<\epsilon$ means that $f(x)$ is in $(L-\epsilon, L+\epsilon)$. Also, we put $0<|x-a|<\delta$ to emphasize that we do not need any information at the value at $x=a$.

Example

Let us prove formally that $\lim_{x \to 0} x^2 = 0$.
We are required to show, given $\epsilon > 0$, that we can find $\delta > 0$ so that

$$|x^2| < \epsilon \text{ whenever } 0 < |x| < \delta.$$

Discovery phase: If we know that $|x^2| < \epsilon$, we can replace $|x^2|$ by the equal number $|x|^2$, and conclude that $|x|^2 < \epsilon$, so $|x| < \sqrt{\epsilon}$. This suggests that $\delta = \sqrt{\epsilon}$ should work for us.

Proof phase: Suppose that we take the value of δ we discovered, $\delta = \sqrt{\epsilon}$. Let us prove that it fits the purpose we have for it. Assume that $0 < |x| < \delta$. Then

$$0 < |x|^2 < (\sqrt{\epsilon})^2, \text{ so } 0 < |x|^2 < \epsilon, \text{ which implies } |x^2| < \epsilon,$$

as required.

Good news

The previous example was there just for your information: proofs like that are frequently done in modules for degrees in maths and theoretical physics, but for our purposes it is usually enough to know that things can be made rigorous.

In fact, from the time when Newton and Leibniz invented most of the differential and integral calculus to the time when mathematicians started dealing with limits rigorously, some 150 years passed when no one worried about things like that.

For our purposes, a good intuitive sense of what having a limit means is usually quite sufficient. We shall formulate a range of theorems about limits to use in applications, but mostly restricting ourselves to intuitive informal proofs.

One-sided limits

A function f is said to *have the limit L as x approaches a from the right* if the values $f(x)$ for $x > a$ and sufficiently close to a are *all* very close to L. In this case, one writes

$$\lim_{x \to a^+} f(x) = L,$$

or

$$f(x) \to L \quad \text{as} \quad x \to a^+.$$

Similarly, a function f is said to *have the limit L as x approaches a from the left* if the values $f(x)$ for $x < a$ and sufficiently close to a are *all* very close to L. In this case, one writes

$$\lim_{x \to a^-} f(x) = L,$$

or

$$f(x) \to L \quad \text{as} \quad x \to a^-.$$

One-sided limits

The most simple example where we need one-sided limits is

$$\mathrm{sign}(x) := \frac{x}{|x|}.$$

Since for all $x > 0$ this function assumes the value 1, and for all $x < 0$ this function assumes the value -1, we have

$$\lim_{x\to 0^-} \mathrm{sign}(x) = -1, \quad \lim_{x\to 0^+} \mathrm{sign}(x) = 1.$$

Of course,

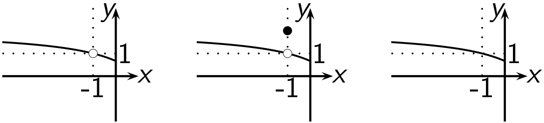
$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

One-sided limits

Let us consider the following three examples.



Note that in each of those examples we have

$$\lim_{x\to -1^-} f(x) = 1 \text{ and } \lim_{x\to -1^+} f(x) = 1,$$

so

$$\lim_{x\to -1} f(x) = 1,$$

and we confirm that the value at $x = a$ is completely irrelevant for the limit $\lim_{x\to a} f(x)$.

Infinite limits

A function f is said to *have the limit* $+\infty$ *as* x *approaches* a *from the right (left)* if the values $f(x)$ for $x > a$ ($x < a$) and sufficiently close to a *all* increase without bound. In this case, one writes

$$\lim_{x \to a^+} f(x) = +\infty \qquad (\lim_{x \to a^-} f(x) = +\infty).$$

If both are true, one writes

$$\lim_{x \to a} f(x) = +\infty.$$

Similarly, a function f is said to *have the limit* $-\infty$ *as* x *approaches* a *from the right (left)* if the values $f(x)$ for $x > a$ ($x < a$) and sufficiently close to a *all* decrease without bound. In this case, one writes

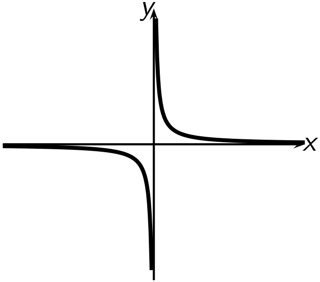
$$\lim_{x \to a^+} f(x) = -\infty \qquad (\lim_{x \to a^-} f(x) = -\infty)$$

If both are true, one writes

$$\lim_{x \to a} f(x) = -\infty.$$

Infinite limits

For example, let us consider the function $f(x) = 1/x$. We plotted its graph earlier in Lecture 4:



In this case, we have

$$\lim_{x\to 0^+} f(x) = +\infty$$

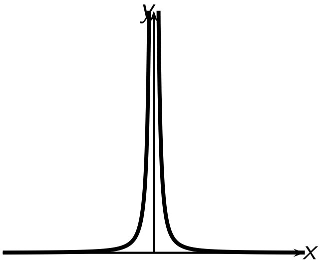
and

$$\lim_{x\to 0^-} f(x) = -\infty,$$

so one sided limits exist and are infinite, but the (two-sided) limit does not exist.

Infinite limits

Now, let us consider the function $f(x) = 1/x^2$. We plotted its graph in Lecture 4 as well:



In this case, we have

$$\lim_{x \to 0^+} f(x) = +\infty$$

and

$$\lim_{x \to 0^-} f(x) = +\infty,$$

so one sided limits exist and are infinite, and also the (two-sided) limit exists, $\lim_{x \to 0} f(x) = +\infty$.