1S11: Calculus for students in Science

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TCD

Lecture 9

Some properties of (finite) limits

- $\lim\limits_{x\to a} c = c$;
- $\lim\limits_{x\to a} x = a$;

Moreover, if the two limits $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist as numbers, then

- $\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$;
- $\lim\limits_{x\to a}(f(x)-g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$;
- $\lim\limits_{x\to a}(f(x)g(x)) = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$;
- in particular, $\lim\limits_{x\to a}(cf(x)) = c \lim\limits_{x\to a} f(x)$;
- $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$ (provided that for even n the limit $\lim\limits_{x\to a} f(x)$ is *positive*);
- $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ whenever $\lim\limits_{x\to a} g(x) \neq 0$.

Examples

$\lim_{x\to 1}(x^3 - 5x + 1) = 1 - 5 + 1 = -3$

$\lim_{y\to 1}(y^3 - 3y + 1)^5 = (1 - 3 + 1)^5 = -1$

...and in general for any polynomial $p(x)$ we have $\lim_{x\to a} p(x) = p(a)$

$\lim_{x\to 2013} \frac{x}{|x|} = \frac{2013}{2013} = 1$

if $\lim_{x\to a} f(x) = 2$, then $\lim_{x\to a} \sqrt[3]{6 + f(x)} = \sqrt[3]{6+2} = 2$

if $\lim_{y\to a} g(y) = 3$, then $\lim_{y\to a} \frac{g(y)+1}{g(y)-2} = \frac{3+1}{3-2} = 4$, since
$\lim_{y\to a}(g(y) - 2) = \lim_{y\to a} g(y) - 2 = 3 - 2 = 1 \neq 0.$

Indeterminate forms

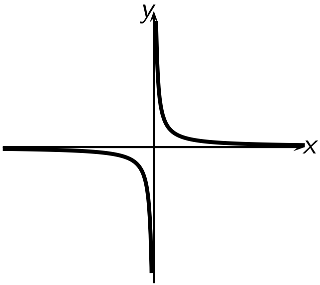
The examples with finite limits we discussed were much simpler than the examples we looked at earlier this week, where some factors cancelled, making undefined expressions defined etc. Limits are particularly useful in cases like that, for example, to determine a limiting value of $\frac{f(x)}{g(x)}$ as $x \to a$, where $f(a) = g(a) = 0$. Such a fraction is called an *indeterminate form of type* $0/0$.

For example $\frac{x^2-4x+4}{x^2-4}$ is an indeterminate form of type $0/0$ at $x = 2$, but not at $x = -2$.

An indeterminate form may have a finite limit, no limit, or an infinite limit.

Infinite limits: reminder

For example, let us consider the function $f(x) = 1/x$. We plotted its graph earlier in Lecture 4:



In this case, we have

$$\lim_{x \to 0^+} f(x) = +\infty$$

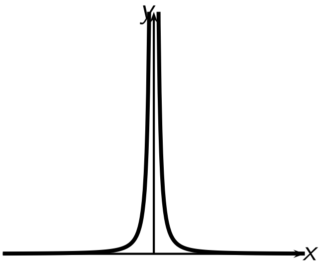
and

$$\lim_{x \to 0^-} f(x) = -\infty,$$

so one sided limits exist and are infinite, but the (two-sided) limit does not exist.

Infinite limits: reminder

Now, let us consider the function $f(x) = 1/x^2$. We plotted its graph in Lecture 4 as well:



In this case, we have

$$\lim_{x \to 0^+} f(x) = +\infty$$

and

$$\lim_{x \to 0^-} f(x) = +\infty,$$

so one sided limits exist and are infinite, and also the (two-sided) limit exists, $\lim_{x \to 0} f(x) = +\infty$.

Indeterminate forms

Theorem. Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function, and let a be a real number.

- If $q(a) \neq 0$ then $f(x)$ is defined for $x = a$, and $\lim\limits_{x \to a} f(x) = f(a)$.
- If $q(a) = 0$ and $p(a) \neq 0$, then $\lim\limits_{x \to a} f(x)$ does not exist as a real number, and may exist or not exist as an infinite limit (depending on the signs of values of $q(x)$ near a: if they are different on the right and on the left, the limit does not exist, if they are the same, there is an infinite limit).
- If $p(a) = q(a) = 0$, then both the numerator and the denominator have a factor $x - a$ (one or more), and to compute the limit $\lim\limits_{x \to a} f(x)$, one brings the fraction to lowest terms and uses the two previous statements.

Indeterminate forms: examples

$\lim\limits_{x\to 1} \frac{x^3-5x+1}{x} = -3$, because the denominator does not vanish

$\lim\limits_{x\to 1} \frac{x+2}{x-1}$ does not exist, since we get $+\infty$ on the right and $-\infty$ on the left

$\lim\limits_{x\to 2} \frac{x-3}{(x-2)^2} = -\infty$ (and does not exist as a number)

$\lim\limits_{x\to 2} \frac{x^2-4x+4}{x^2-4} = \lim\limits_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = 0$

$\lim\limits_{x\to 3} \frac{x^2-4x+3}{x^2-6x+9} = \lim\limits_{x\to 3} \frac{(x-1)(x-3)}{(x-3)^2}$, does not exist

$\lim\limits_{x\to 1} \frac{x^2-4x+3}{x^2-3x+2} = \lim\limits_{x\to 1} \frac{(x-1)(x-3)}{(x-1)(x-2)} = 2$

Infinite limits

In terms of infinite limits we can make the notion of a vertical asymptote more precise:

The graph of a function f has the line $x = a$ as a vertical asymptote if at least one of the four following situations occur:

$$\lim_{x\to a^+} f(x) = +\infty, \lim_{x\to a^-} f(x) = +\infty, \lim_{x\to a^+} f(x) = -\infty, \lim_{x\to a^-} f(x) = -\infty.$$

E.g., for $f(x) = 1/x$ and $a = 0$, the first and the last formulas apply, and for $f(x) = 1/x^2$ and $a = 0$, the first and the second formulas apply.

Question. Is it possible to find a function for which both the first and the third formulas apply? No: these formulas contradict each other.

Exercise. Sketch examples of graphs for which exactly one of those situations occurs.