MA1S11 (Dotsenko) Sample questions and answers for the calculus part of 1S11

Michaelmas 2013

1. Compute the limit $\lim_{x\to 0}\frac{\tan(7x)}{e^{3x}-1}$.

   Solution. We have

$$\lim_{x\to 0}\frac{\tan(7x)}{e^{3x}-1}=\frac{7}{3}\lim_{x\to 0}\left(\frac{\sin(7x)}{7x}\frac{1}{\cos(7x)}\frac{3x}{e^{3x}-1}\right)=$$
$$=\frac{7}{3}\lim_{x\to 0}\frac{\sin(7x)}{7x}\lim_{x\to 0}\frac{1}{\cos(7x)}\lim_{x\to 0}\frac{3x}{e^{3x}-1}=\frac{7}{3}\lim_{t=7x\to 0}\frac{\sin t}{t}\cdot\lim_{u=7x\to 0}\frac{1}{\cos u}\cdot\lim_{v=3x\to 0}\frac{v}{e^v-1}=$$
$$=\frac{7}{3}\lim_{t\to 0}\frac{\sin t}{t}\cdot\frac{1}{\lim\limits_{u\to 0}\cos u}\cdot\frac{1}{\lim\limits_{v\to 0}\frac{e^v-1}{v}}=\frac{7}{3}\cdot 1\cdot 1\cdot 1=\frac{7}{3}.$$

2. From the first principles, prove that the derivative of the function $f(x)=\frac{1}{\sqrt{x}}$ is given by the formula $\frac{-1}{2x\sqrt{x}}$.

   Solution. We have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}}=$$
$$=\lim_{h\to 0}\frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}=\lim_{h\to 0}\frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}=$$
$$=\lim_{h\to 0}\frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}=\frac{-1}{\sqrt{x}\sqrt{x}\cdot 2\sqrt{x}}=\frac{-1}{2x\sqrt{x}}.$$

3. Is the function

$$f(x)=\begin{cases}x^3\sin\frac{1}{x}, x\ne 0,\\ 0, x=0\end{cases}$$

   continuous at $x=0$? differentiable at $x=0$? twice differentiable at $x=0$? Explain your answer.

   Solution. Since $-1\le\sin\frac{1}{x}\le 1$, we have $-|x^3|\le x^3\sin\frac{1}{x}\le|x^3|$, and therefore by Squeezing Theorem

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}x^3\sin\frac{1}{x}=0=f(0),$$

   so f is continuous. Moreover, since we have

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{x^3\sin\frac{1}{x}}{x}=\lim_{x\to 0}x^2\sin\frac{1}{x},$$

   we may apply Squeezing Theorem again to conclude that $f'(0)$ exists and is equal to 0. For $x\ne 0$, the product rule and the chain rule give us

$$f'(x)=3x^2\sin\frac{1}{x}-x\cos\frac{1}{x}.$$

   This latter formula we use to compute the second derivative of f at $x=0$:

$$f''(0)=\lim_{x\to 0}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0}\frac{3x^2\sin\frac{1}{x}-x\cos\frac{1}{x}}{x}=\lim_{x\to 0}\left(3x\sin\frac{1}{x}-\cos\frac{1}{x}\right).$$

From this formula, and the fact that by Squeezing Theorem the limit $\lim_{x\to 0} 3x\sin\frac{1}{x}$ exists and is equal to 0, we conclude that $f''(0)$ is not defined, since $\cos\frac{1}{x}$ has no limit as x approaches 0. (For instance, it takes both the value 1 and the value -1 at points arbitrarily close to 0.)

4. Compute the derivatives:

   (a) $(\tan(7+5\ln x))^3$; (b) $\cos^{-1} x$; (c) $x^{1/x}$; (d) $\ln\left(\frac{e^x}{1+e^x}\right)$.

   Solution.

   (a) Applying the chain rule a few times, we get

$$\left((\tan(7+5\ln x))^3\right)' = 3(\tan(7+5\ln x))^2 \frac{1}{\cos^2(7+5\ln x)} \frac{5}{x} = 15 \frac{\sin^2(7+5\ln x)}{x\cos^4(7+5\ln x)}.$$

   (b) Let us apply the chain rule to $\cos(\cos^{-1} x) = x$:

$$-\sin(\cos^{-1} x)\cdot(\cos^{-1} x)' = 1,$$

   so

$$(\cos^{-1} x)' = -\frac{1}{\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-(\cos(\cos^{-1} x))^2}} = -\frac{1}{\sqrt{1-x^2}}.$$

   (c) Rewriting $x^{1/x} = e^{\ln x^{1/x}} = e^{\frac{\ln x}{x}}$, and applying the chain rule, we get

$$(x^{1/x})' = e^{\frac{\ln x}{x}} \cdot \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = x^{1/x}\frac{1-\ln x}{x^2}.$$

   (d) Applying the chain rule, we get

$$\left(\ln\left(\frac{e^x}{1+e^x}\right)\right)' = \frac{1+e^x}{e^x}\cdot\frac{e^x(1+e^x)-e^x\cdot e^x}{(1+e^x)^2} = \frac{1}{1+e^x}.$$

5. Compute $f'(\pi/6)$, if $f(x) = \tan^{-1}(\cos x)$.

   Solution. Note that by chain rule we have $\frac{1}{\cos^2(\tan^{-1} x)}\cdot(\tan^{-1} x)' = 1$, so $(\tan^{-1} x)' = \cos^2(\tan^{-1} x)$. Since $\sin^2(\tan^{-1} x) + \cos^2(\tan^{-1} x) = 1$, we have $\tan^2(\tan^{-1} x) + 1 = \frac{1}{\cos^2(\tan^{-1} x)}$, and $\cos^2(\tan^{-1} x) = \frac{1}{1+x^2}$. With that in mind, we compute

$$f'(x) = \frac{1}{1+\cos^2 x}\cdot(-\sin x).$$

   Substituting $x = \pi/6$, we get $f'(\pi/6) = \frac{1}{1+3/4}\cdot(-1/2) = -\frac{2}{7}$.

6. Compute $f'(e)$ for $f(x) = \frac{x^3}{\ln x}$.

   Solution. We have

$$f'(x) = \frac{3x^2\ln x - x^3\frac{1}{x}}{\ln^2 x},$$

   so

$$f'(e) = \frac{3e^2 - e^2}{1} = 2e^2.$$

7. "The slope of the tangent to the curve $y = ax^3 + bx + 4$ at the point $(2, 14)$ on that curve is 21." Find the values of a and b for which it is true.

   Solution. Since the point $(2, 14)$ is on the curve, we have $14 = 8a + 2b + 4$, or $4a + b = 5$. Since the slope is 21, we have $y'(x) = 3ax^2 + b$ is 21 when $x = 2$, so $12a + b = 21$. Subtracting these equations, we get $8a = 16$, so $a = 2$, which implies $b = -3$.

8. For $f(x) = \sin(\ln x)$, show that $x^2 f'' + x f' + f = 0$.

   Solution. We have $f'(x) = \cos(\ln x) \cdot \frac{1}{x}$ by chain rule. Furthermore, by chain rule and product rule, we have

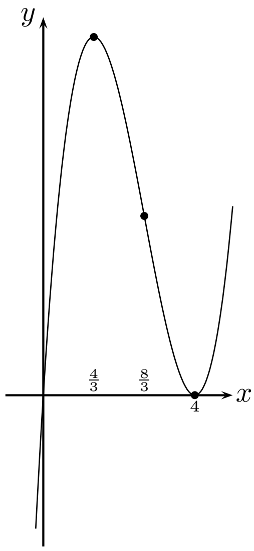
$$f''(x) = -\sin(\ln x) \cdot \frac{1}{x} \cdot \frac{1}{x} + \cos(\ln x) \cdot \left(-\frac{1}{x^2}\right).$$

   This means that

$$x^2 f'' + x f' + f = -\sin(\ln x) - \cos(\ln x) + \cos(\ln x) + \sin(\ln x) = 0.$$

9. Determine relative extrema and inflection points of the graph $y = x^3 - 8x^2 + 16x$, and draw a rough sketch of that graph.

   Solution. First of all, we have $y = x^3 - 8x^2 + 16x = x(x^2 - 8x + 16) = x(x-4)^2$, so the x-intercepts are $x = 0$ and $x = 4$. Next, we have $y'(x) = 3x^2 - 16x + 16 = (x-4)(3x-4)$, so the relative extrema are at $x = 4$ and $x = 4/3$. Moreover, since $y'(x)$ changes from positive to negative at $4/3$, at that point a relative maximum is attained, and since $y'(x)$ changes from negative to positive at $x = 4$, at that point a relative minimum is attained. Finally, $y''(x) = 6x - 16$, so the only inflection point is $x = 8/3$, where the graph is changes from concave down ($y''(x) < 0$) to concave up ($y''(x) > 0$)). Using all this information, we obtain the following graph:



10. Show that among all the rectangles of area A, the square has the minimum perimeter.

   Solution. Suppose that one of the sides of the rectangle is x, so that the other one is $\frac{A}{x}$. Then the perimeter of the rectangle is $2x + \frac{2A}{x}$. To find the minimum of this function (with the domain being the open ray $(0, +\infty)$ from the context), we should examine the points where the derivative vanishes. The derivative of this function is

$$2 - \frac{2A}{x^2},$$

   and it vanishes precisely for $x^2 = A$, so the only solution in the domain of our function is $x = \sqrt{A}$ (which precisely corresponds to the situation when the rectangle is a square). The second derivative of this function is $\frac{4A}{x^3}$, which is positive everywhere where f is defined, so $x = \sqrt{A}$ is a local minimum. It is also an absolute minimum, since as $x \to 0$ or $x \to +\infty$ the limit of the perimeter is $+\infty$.

11. The concentration C of an antibiotic in the bloodstream after time t is given by

$$C = \frac{5t}{1 + \frac{t^2}{k^2}}$$

for a certain constant k. If it is known that the maximal concentration is reached at $t = 6$ hours, find the value of k.

Solution. To find the maximum of this function (with the domain being the closed ray $[0, +\infty)$ from the context), we should examine the points where the derivative vanishes. The derivative of this function is

$$\frac{5(1 + \frac{t^2}{k^2}) - 5t \cdot \frac{2t}{k^2}}{\left(1 + \frac{t^2}{k^2}\right)^2} = 5\frac{1 - \frac{t^2}{k^2}}{\left(1 + \frac{t^2}{k^2}\right)^2},$$

so in the domain of our function the only point where it vanishes is $t = k$. Also, for $t < k$ the derivative is positive, and for $t > k$ it is negative, so it is indeed a point where the function reaches its maximal value. We conclude that $k = 6$.

12. Evaluate the integrals

(a) $\int \frac{\sin 2\theta}{1+\cos 2\theta}\, d\theta$; (b) $\int \frac{x\,dx}{1+x^2}$; (c) $\int x^3 \sqrt[3]{1-4x}\, dx$.

Solution.

(a) We use u-substitution with $u = 1 + \cos 2\theta$:

$$\int \frac{\sin 2\theta}{1 + \cos 2\theta}\, d\theta = -\frac{1}{2}\int \frac{d(1 + \cos 2\theta)}{1 + \cos 2\theta} = -\frac{1}{2}\ln(1 + \cos 2\theta) + C.$$

(b) We use u-substitution with $u = 1 + x^2$:

$$\int \frac{x\,dx}{1 + x^2} = \frac{1}{2}\int \frac{d(1 + x^2)}{1 + x^2} = \frac{1}{2}\ln(1 + x^2) + C.$$

(c) We use u-substitution with $u = 1 - 4x$

$$\int x^3 \sqrt[3]{1 - 4x}\, dx = -\frac{1}{4}\int \left(\frac{1-u}{4}\right)^3 \sqrt[3]{u}\, du = -\frac{1}{256}\int (1 - 3u + 3u^2 - u^3)u^{1/3}]\, du =$$

$$= -\frac{1}{256}\left(\frac{3}{4}u^{4/3} - \frac{9}{7}u^{7/3} + \frac{9}{10}u^{10/3} - \frac{3}{13}u^{13/3}\right) + C =$$

$$= -\frac{1}{256}\left(\frac{3}{4}(1 - 4x)^{4/3} - \frac{9}{7}(1 - 4x)^{7/3} + \frac{9}{10}(1 - 4x)^{10/3} - \frac{3}{13}(1 - 4x)^{13/3}\right) + C.$$

13. Evaluate the integrals

(a) $\int\limits_{1/2}^{1} \frac{3}{2x}\, dx$; (b) $\int\limits_{0}^{\pi} \frac{\cos^2 x}{1+\sin x}\, dx$; (c) $\int\limits_{e^{-1}}^{e} \frac{\sqrt{1-(\ln x)^2}}{x}\, dx$.

Solution.

(a) Since $F(x) = \frac{3}{2}\ln x$ is an antiderivative of $\frac{3}{2x}$, we have

$$\int\limits_{1/2}^{1} \frac{3}{2x}\, dx = \frac{3}{2}\ln 1 - \frac{3}{2}\ln\left(\frac{1}{2}\right) = \frac{3}{2}\ln 2.$$

(b) Since $\cos^2 x = 1 - \sin^2 x = (1 - \sin x)(1 + \sin x)$, we have

$$\int_0^{\pi} \frac{\cos^2 x}{1 + \sin x}\, dx = \int_0^{\pi} (1 - \sin x)\, dx = (x + \cos x)]_0^{\pi} = (\pi - 1) - (0 + 1) = \pi - 2.$$

(c) Using u-substitution with $u = \ln x$, so that $du = \frac{dx}{x}$, we have

$$\int_{e^{-1}}^{e} \frac{\sqrt{1 - (\ln x)^2}}{x}\, dx = \int_{-1}^{1} \sqrt{1 - u^2}\, du = \frac{\pi}{2},$$

the last equality coming from the fact that $\int_{-1}^{1} \sqrt{1 - u^2}\, du$ is the area of the half-circle $0 \leq y \leq \sqrt{1 - u^2}$ of radius 1.

14. Find a positive value of k for which the area under the graph of $y = e^{3x}$ over the interval $[0, k]$ is 11 square units.

Solution. We have

$$\int_0^k e^{3x}\, dx = \frac{1}{3}\int_0^k d(e^{3x}) = \frac{1}{3}(e^{3k} - 1),$$

and this quantity is equal to 11 when $e^{3k} = 34$, so $k = \frac{1}{3}\ln(34)$.

15. Compute the area of the region between the graphs $y = xe^x$ and $y = x^2 e^x$.

Solution. First let us find the points where these graphs meet:

$$xe^x = x^2 e^x$$

has solutions $x = 0$ and $x = 1$. Between these values of x we have $xe^x \geq x^2 e^x$, so the area in question is

$$\int_0^1 (xe^x - x^2 e^x)\, dx = \int_0^1 (x - x^2)\, d(e^x) = (x - x^2)e^x]_0^1 - \int_0^1 e^x(1 - 2x)\, dx =$$

$$= -\int_0^1 (1 - 2x)\, d(e^x) = -(1 - 2x)e^x]_0^1 - \int_0^1 2e^x\, dx = e + 1 - 2(e - 1) = 3 - e.$$