MA1S11 (Dotsenko) Tutorial/Exercise Sheet 4

Week 5, Michaelmas 2013

Please hand in your work in the end of the tutorial. Make sure you put your name and student ID number on what you hand in.

A complete solution to every question is worth 2 marks.

Reminder:

1. For limits of rational functions at infinity, use the formula

$$f(x) = a_0 + a_1x + \cdots + a_nx^n = a_nx^n\left(\frac{a_0}{a_nx^n} + \frac{a_1}{a_nx^{n-1}} + \cdots + \frac{a_{n-1}}{a_nx} + 1\right)$$

in both the numerator and the denominator to compute limits.

2. For limits of differences $\sqrt{f(x)} - \sqrt{g(x)}$ (or simply $\sqrt{f(x)} - h(x)$), it is often useful to apply the formula $a - b = \frac{a^2-b^2}{a+b}$.

3. (The Squeezing Theorem) Suppose that we have three functions $f(x)$, $g(x)$, and $h(x)$, and that we can prove the inequalities

$$g(x) \leq f(x) \leq h(x)$$

for all x in some open interval containing the number c, possibly with the exception of c itself. Then if g and h have the same limit L at c, then f also has the limit L at c.

4. We have $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

Questions

1. Which of the following limits exist (as finite or infinite limits)? Explain your answer and compute them.
   - $\lim\limits_{x\to+\infty} \frac{3-5x^3}{1+4x+x^3}$.
   - $\lim\limits_{x\to-\infty} \frac{3-5x^3}{1+4x+x^3}$.

2. Which of the following limits exist (as finite or infinite limits)? Explain your answer and compute them.
   - $\lim\limits_{x\to-\infty} \frac{\sqrt{x^2+x}}{3x-1}$.
   - $\lim\limits_{x\to+\infty} \left(\sqrt{x+1} - \sqrt{x}\right)$.

3. Which of the following limits exist (as finite or infinite limits)? Explain your answer and compute them.

- $\lim\limits_{x\to 0} \frac{1-\cos x}{\sin x}$.
- $\lim\limits_{x\to 0} \frac{\tan x - \sin x}{x^3}$.

4. Show that
$$-|x| \leq x \sin \frac{1}{x} \leq |x|$$
for all $x \neq 0$, and explain why $\lim\limits_{x\to 0} x \sin \frac{1}{x} = 0$.

5. Let us consider function $f(x) = \begin{cases} \sin \frac{1}{x}, x \neq 0, \\ a, \quad x = 0. \end{cases}$ Does there exist a choice of a for which this function is continuous at $x = 0$? Explain your answer.