Linear Algebra (MA113): solutions to the final exam

1. Solution: (a) Computing minors and cofactors explicitly, we get: $M_{11} = -3$, $M_{12} = -3$, $M_{13} = 0$, $M_{21} = -3$, $M_{22} = -6$, $M_{23} = -3$, $M_{31} = 0$, $M_{32} = -3$, $M_{33} = -2$, $C_{11} = -3$, $C_{12} = 3$, $C_{13} = 0$, $C_{21} = 3$, $C_{22} = -6$, $C_{23} = 3$, $C_{31} = 0$, $C_{32} = 3$, $C_{33} = -2$, $\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 3$ and $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2/3 \end{pmatrix}$.

   (b) We have $x = A^{-1}b = \begin{pmatrix} -6 \\ 9 \\ -7/3 \end{pmatrix}$.

2. Solution: since all numbers from 1 to 7 should occur as first subscripts (an element from each row is present), we should have $\{i, j\} = \{5, 7\}$; the same is true for columns, so we have $\{k, l\} = \{2, 6\}$. Let us consider the case $i = 5$, $j = 7$, $k = 2$, $l = 6$: in this case the product is $a_{14}a_{23}a_{35}a_{42}a_{56}a_{67}a_{71}$, the corresponding permutation $4, 3, 5, 2, 6, 7, 1$ contains 10 inversions, and the coefficient is equal to 1. Since a transposition of rows/columns changes all signs in the expansion of the determinant, we see that for $i = 7$, $j = 5$, $k = 2$, $l = 6$ and $i = 5$, $j = 7$, $k = 6$, $l = 2$ the coefficient is -1, and for $i = 7$, $j = 5$, $k = 6$, $l = 2$ the coefficient is 1. So the answer is $i = 7$, $j = 5$, $k = 2$, $l = 6$ or $i = 5$, $j = 7$, $k = 6$, $l = 2$.

3. Solution: (a) We have
$$\operatorname{tr}(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{i,j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \operatorname{tr}(BA).$$

   (b) From the previous statement, it follows that $\operatorname{tr}(ABC) = \operatorname{tr}(CAB) = \operatorname{tr}(BCA)$ and $\operatorname{tr}(ACB) = \operatorname{tr}(BAC) = \operatorname{tr}(CBA)$, so either all the six traces are equal or there are two distinct numbers. If $A = B = C$, all the traces are equal, and for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, We have $ABC = C$ and $BAC = 0$, so $\operatorname{tr}(ABC) = 1$, $\operatorname{tr}(BAC) = 0$ and there are two distinct numbers.

4. Solution: denote these vectors by v_1 and v_2; we have $Av_1 = \begin{pmatrix} -7 \\ 0 \\ 21 \\ -7 \end{pmatrix}$ and $Av_2 = \begin{pmatrix} -1 \\ -4 \\ -25 \\ 11 \end{pmatrix}$. Our subspace U is invariant if the image of every vector is again in U; it is enough to check that the images of v_1 and v_2 are in U, so we have to find out whether or not Av_i can be represented as combinations of v_1 and v_2. Solving the

corresponding systems of equations, we get $Av_1 = 7v_1 + 7v_2$, $Av_2 = -7v_1 - 3v_2$, so U is invariant.

5. Solution: (a) The characteristic polynomial of B is $(t-2)^2$, so the only eigenvalue is 2. We have $B - 2I \neq 0$, $(B-2I)^2 = 0$, $\dim \operatorname{Ker}(B-2I) = 1$, $\dim \operatorname{Ker}(B-2I)^2 = 2$, so to obtain the Jordan basis of B we should take a basis of $\operatorname{Ker}(B-2I)^2$ relative to $\operatorname{Ker}(B-2I)$, for which we can take, for example, $f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; the two vectors $e_1 = (B-2I)f = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $e_2 = f$ form a Jordan basis for B, the transition matrix $C = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ is obtained by joining the vectors of the Jordan basis together, and the Jordan normal form of B is $J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Indeed, $(B-2I)^2 f = 0$, so $Be_1 = B(B-2I)f = 2I(B-2I)f = 2(B-2I)f = 2e_1$ and $Be_2 = Bf = (B-2I)f + 2If = (B-2I)f + 2f = e_1 + 2e_2$.

   (b) We have
$$B^n = CJ^nC^{-1} = \begin{pmatrix} 2^n - n2^{n-1} & -n2^{n-1} \\ n2^{n-1} & 2^n + n2^{n-1} \end{pmatrix}.$$
   Also, if we let $v_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, we get $v_{k+1} = Bv_k$, so by induction $v_n = B^n v_0 = \begin{pmatrix} 2^n + n2^{n+1} \\ -(3n+5)2^n \end{pmatrix}$.

6. Solution: (a) A basis $e_1, \ldots, e_n$ of a Euclidean space V is called orthogonal if $(e_i, e_j) = 0$ for all $i \neq j$. An orthogonal basis is called orthonormal if $(e_i, e_i) = 1$ for all i.

   (b) In class, we proved that n vectors in $\mathbb{R}^n$ form a basis if and only if they are linearly independent. For n columns, they are linearly independent if and only if the system $Ax = 0$ has only the trivial solution, where A is the matrix whose columns are our given columns. In our case, $\det(A) = 15 \neq 0$, so, as we proved in class, A is invertible, and the system $Ax = 0$ has only the trivial solution.

   (c) We have
$$\begin{aligned} e_1 &= f_1, \\ e_2 &= f_2 - \frac{(f_2, e_1)}{(e_1, e_1)} e_1, \\ e_3 &= f_3 - \frac{(f_3, e_1)}{(e_1, e_1)} e_1 - \frac{(f_3, e_2)}{(e_2, e_2)} e_2, \end{aligned}$$
   so
$$e_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, e_2 = 1/5 \begin{pmatrix} 6 \\ -10 \\ 3 \end{pmatrix}, e_3 = 15/29 \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}.$$

7. Solution: (a) For a real vector space V, a function $f\colon V\times V\to\mathbb{R}$ is called a bilinear form if

$$\begin{aligned} f(cv,w) &= cf(v,w),\\ f(v,cw) &= cf(v,w),\\ f(v_1+v_2,w) &= f(v_1,w)+f(v_2,w),\\ f(v,w_1+w_2) &= f(v,w_1)+f(v,w_2) \end{aligned}$$

for all $v,w,v_1,v_2,w_1,w_2\in V$, $c\in\mathbb{R}$. A symmetric bilinear form is said to be positive definite if $f(v,v)\ge 0$ for all $v\in V$, and $f(v,v)=0$ only for $v=0$.

(b) It is easy to see that relative to the standard basis $1,t,t^2$ of V the matrix of our bilinear form is

$$\begin{pmatrix} -2a & 2/3 & -2a/3\\ 2/3 & -2a/3 & 2/5\\ -2a/3 & 2/5 & -2a/5 \end{pmatrix}.$$

In class, we proved the Sylvester's criterion stating that a quadratic form is positive definite if and only if the principal minors $\Delta_1=a_{11}$, $\Delta_2=\det\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$, ..., Δ_n of its matrix relative to any basis are all positive. In our case, we have $\Delta_1=-2a$, $\Delta_2=4(a^2/3-1/9)$, $\Delta_3=8(-4/135a^3+4/225a)$, so we have $a<0$, $a^2<1/3$, $a^2<3/5$. The common solution set to all these inequalities is the set $\{a\colon a<-\sqrt{3/5}\}$.