Solutions to the sample final paper

1. Since we are asked to solve both the homogeneous and the inhomogeneous system, it makes sense to bring the augmented matrix to the reduced row echelon form:

$$\begin{pmatrix} 4 & 0 & 1 & 2 & 1 & | & 7 \\ 3 & 1 & 2 & 0 & -1 & | & 2 \\ 5 & -1 & 0 & 4 & 2 & | & 0 \end{pmatrix}$$

$$\overset{1/4(1),(2)-3(1),(3)-5(1)}{\mapsto} \begin{pmatrix} 1 & 0 & 1/4 & 1/2 & 1/4 & | & 7/4 \\ 0 & 1 & 5/4 & -3/2 & -7/4 & | & -13/4 \\ 0 & -1 & -5/4 & 3/2 & 3/4 & | & -35/4 \end{pmatrix}$$

$$\overset{(3)+(2)}{\mapsto} \begin{pmatrix} 1 & 0 & 1/4 & 1/2 & 1/4 & | & 7/4 \\ 0 & 1 & 5/4 & -3/2 & -7/4 & | & -13/4 \\ 0 & 0 & 0 & 0 & -1 & | & -12 \end{pmatrix}$$

$$\overset{-1\cdot(3),(1)-1/4(3),(2)+7/4(3)}{\mapsto} \begin{pmatrix} 1 & 0 & 1/4 & 1/2 & 0 & | & -5/4 \\ 0 & 1 & 5/4 & -3/2 & 0 & | & 71/4 \\ 0 & 0 & 0 & 0 & 1 & | & 12 \end{pmatrix}$$

It follows that the reduced row echelon form of the original matrix is

$$\begin{pmatrix} 1 & 0 & 1/4 & 1/2 & 0 \\ 0 & 1 & 5/4 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

variables x_1, x_2, and x_5 are leading, variables x_3 and x_4 are free, the general solution to the homogeneous system $Ax = 0$ is

$$(-\frac{1}{4}u - \frac{1}{2}v, -\frac{5}{4}u + \frac{3}{2}v, u, v, 0),$$

where u and v are arbitrary parameters, and the general solution to the given inhomogeneous system is

$$(-\frac{5}{4} - \frac{1}{4}u - \frac{1}{2}v, \frac{71}{4} - \frac{5}{4}u + \frac{3}{2}v, u, v, 12),$$

where u and v are arbitrary parameters.

2. In order for this term to occur in the expansion at all, we should have

$$\{4, 3, 6, j, i, 1, 7\} = \{1, l, 4, 5, 6, k, 3\} = \{1, 2, 3, 4, 5, 6, 7\}$$

(one element should be taken from each row and from each column). This implies $\{i, j\} = \{2, 5\}$ and $\{k, l\} = \{2, 7\}$, so there are four different options for i, j, k, l. Let us consider the case $i = k = 2$, $j = 5$, $l = 7$, that is the term

$$a_{41}a_{3l}a_{64}a_{j5}a_{i6}a_{1k}a_{73} = a_{12}a_{26}a_{37}a_{41}a_{55}a_{64}a_{73}.$$

The corresponding permutation 2 6 7 1 5 4 3 has 12 inversions and hence is even. The term for $i = 2$, $k = 7$, $j = 5$, $l = 2$ is obtained from this one by exchanging two columns, so the corresponding permutation is odd, the term for $i = 5$, $k = 2$, $j = 2$, $l = 7$ is obtained by exchanging two rows, so the permutation is also odd, and finally the term for $i = 5$, $k = 7$, $j = 2$, $l = 2$ is obtained by exchanging two rows and then two columns; each of these operations changes the parity, so the permutation is again even. Overall, there are two options: $i = 2$, $k = 7$, $j = 5$, $l = 2$ and $i = 5$, $k = 2$, $j = 2$, $l = 7$.

3. Assume the A is invertible. Then we have

$$B - A^{-1}BA = I,$$

so $0 = \mathrm{tr}(B) - \mathrm{tr}(A^{-1}BA) = \mathrm{tr}(B - A^{-1}BA) = \mathrm{tr}(I) = n$, which is a contradiction.

4. Let us first find the bases of these subspaces. To make our computation somehow more familiar, we write coordinates of the vectors in rows. To find a basis of the space spanned by rows of a matrix, we bring it to its reduced row echelon form; the nonzero rows of the result give us a basis. Doing that for our systems of vectors, we have

$$\begin{pmatrix} 2 & 1 & 0 & -4 & 2 \\ -4 & 1 & 3 & -1 & 2 \\ 0 & 5 & -1 & -1 & 14 \end{pmatrix} \overset{1/2(1),(2)+4(1)}{\longmapsto} \begin{pmatrix} 1 & 1/2 & 0 & -2 & 1 \\ 0 & 3 & 3 & -9 & 6 \\ 0 & 5 & -1 & -1 & 14 \end{pmatrix} \overset{1/3(2),(3)-5(2)}{\longmapsto} \begin{pmatrix} 1 & 1/2 & 0 & -2 & 1 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 0 & -16 & 44 & 16 \end{pmatrix}$$
$$\overset{1-1/2(2),-1/16(3)}{\longmapsto} \begin{pmatrix} 1 & 0 & -1/2 & -1/2 & 0 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 0 & 1 & -11/4 & 1 \end{pmatrix} \overset{(2)-(3),1+1/2(3)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & -15/8 & 1/2 \\ 0 & 1 & 0 & -1/4 & 1 \\ 0 & 0 & 1 & -11/4 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 1 \\ 2 & -1 & -2 & -3 & -1 \\ 1 & 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix} \overset{1/2(1),(2)-2(1),(3)-(1)}{\longmapsto} \begin{pmatrix} 1 & 1/2 & 0 & 1/2 & 1/2 \\ 0 & -2 & -2 & -4 & -2 \\ 0 & -1/2 & -2 & -5/2 & 3/2 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix}$$
$$\overset{-1/2(2),(1)-1/2(2),(3)+1/2(2),(4)-(2)}{\longmapsto} \begin{pmatrix} 1 & 0 & -1/2 & -1/2 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & -3/2 & -3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{-2/3(3),(1)+1/2(3),(2)-(3)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1 & 7/3 \\ 0 & 0 & 1 & 1 & -4/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This means that each of the subspaces is three-dimensional. To compute the intersection, recall that by definition the intersection consists of all vectors that belong to both of the subspaces. Let us denote by u_1, u_2, u_3 the basis vectors for U found above, and by w_1, w_2, w_3 the basis vectors for W found above. Then the intersection consists of all vectors v that can be represented in the form

$$v = c_1u_1 + c_2u_2 + c_3u_3 = c_4w_1 + c_5w_2 + c_6w_3$$

for some $c_1, c_2, c_3, c_4, c_5, c_6$, or, equivalently,

$$c_1u_1 + c_2u_2 + c_3u_3 - c_4w_1 - c_5w_2 - c_6w_3 = 0.$$

This is a homogeneous system of linear equations with unknowns c_i. Its matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -15/8 & -1/4 & -11/4 & 0 & 1 & 1 \\ 1/2 & 1 & 1 & -2/3 & 7/3 & -4/3 \end{pmatrix}$$

Bringing it to the reduced row echelon form (calculations are omitted), we get the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

so the general solution is $c_1 = -2t$, $c_2 = 0$, $c_3 = t$, $c_4 = -2t$, $c_5 = 0$, $c_6 = t$, and the intersection can be described as the set of all vectors of the form

$$t(-2u_1 + u_3) = t(-2w_1 + w_3),$$

so for a basis of the intersection we can take $-2u_1 + u_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

5. (a) The characteristic polynomial of our matrix is $t^2 - 3t + 1$, so the eigenvalues are $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{3-\sqrt{5}}{2}$. Solving the systems for the eigenvectors, we obtain the corresponding eigenvectors $t\begin{pmatrix} 1 \\ \frac{3\pm\sqrt{5}}{2} \end{pmatrix}$.

   (b) Our matrix has a basis of eigenvectors (there are two linearly independent ones, and they have to form a basis), so the Jordan normal form is $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. For the transition matrix we take the matrix whose columns are aigenvectors: $C = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$.

(c) We have $C^{-1}BC = J$, so $B^n = CJ^nC^{-1}$. One can easily see that $C^{-1} = \frac{1}{\lambda_2-\lambda_1}\begin{pmatrix}\lambda_2 & -1\\ -\lambda_1 & 1\end{pmatrix}$, so (we use the formula $\lambda_1\lambda_2 = 1$)

$$B^n = \begin{pmatrix} \frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_2-\lambda_1} & \frac{\lambda_2^n-\lambda_1^n}{\lambda_2-\lambda_1} \\ \frac{\lambda_1^n-\lambda_2^n}{\lambda_2-\lambda_1} & \frac{\lambda_2^{n+1}-\lambda_1^{n+1}}{\lambda_2-\lambda_1} \end{pmatrix}.$$

Let us consider the sequence of vectors $v_n = \begin{pmatrix} x_n \\ x_{n+1}\end{pmatrix}$. Clearly, $v_{n+1} = Bv_n$ which, by induction, means that

$$v_n = B^n v_0 = B^n \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} \frac{\lambda_2^n-\lambda_1^n}{\lambda_2-\lambda_1} \\ \frac{\lambda_2^{n+1}-\lambda_1^{n+1}}{\lambda_2-\lambda_1}\end{pmatrix}.$$

Finally, x_n is the first coordinate of v_n, that is $\frac{\lambda_2^n-\lambda_1^n}{\lambda_2-\lambda_1}$.

6. (a) A basis $e_1, \ldots, e_n$ of a Euclidean space is called orthogonal if $(e_i, e_j) = 0$ for $i \neq j$ and $(e_i, e_) = 1$ for all i.

(b) The matrix of coordinates of these vectors relative to the standard basis 1, t, t^2 is

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix},$$

its determinant is equal to $1 \neq 0$, so it is invertible and is a transition matrix to a basis.

(c) We have

$$e_1 = f_1,$$
$$e_2 = f_2 - \frac{(e_1, f_2)}{(e_1, e_1)}e_1,$$
$$e_3 = f_3 - \frac{(e_1, f_3)}{(e_1, e_1)}e_1 - \frac{(e_2, f_3)}{(e_2, e_2)}e_2,$$

so

$$e_1 = 1 + t,$$
$$e_2 = 1 + 2t + t^2 - \frac{4}{8/3}(1+t) = -\frac{1}{2} + \frac{1}{2}t + t^2,$$
$$e_3 = 2 - t^2 - \frac{10/3}{8/3}(1+t) - \frac{-11/15}{2/5}(-\frac{1}{2} + \frac{1}{2}t + t^2) = \frac{5}{6}t^2 - \frac{1}{3}t - \frac{1}{6}.$$

7. (a) For a vector space V, a function $f\colon V \times V \to \mathbb{R}$ is said to define an inner product if it is bilinear ($f(v_1+v_2, w) = f(v_1, w) + f(v_2, w)$, ($f(v, w_1+w_2) = f(v, w_1) + f(v, w_2)$, $f(cv, w) = f(v, cw) = cf(v, w)$ for all v, w, v_1, v_2, w_1, w_2), symmetric ($f(v, w) = f(w, v)$ for all v, w), and positive definite ($f(v, v) \geq 0$ for all v, amd $f(v, v) = 0$ only for $v = 0$).

(b) This function is clearly a bilinear form (bilinear and symmetric); its matrix relative to the given basis is

$$\begin{pmatrix} 1 & a-1 & a+1 \\ a-1 & 1 & 4 \\ a+1 & 4 & 20 \end{pmatrix}.$$

According to the Sylvester's criterion, to be positive definite, this matrix should have its top left corner minors to be positive. They are

$$\Delta_1 = 1, \Delta_2 = 1 - (a-1)^2 = -a(a-2), \Delta_3 = -13a^2 + 38a - 25 = -(a-1)(13a-25).$$

To have these expression positive, we should have

$$0 < a < 2, 1 < a < 25/13,$$

which together mean $1 < a < 25/13$.