Orthonormal bases, orthogonal complements, and orthogonal direct sums

A sequence of vectors $e_1, \ldots, e_n$ of a n-dimensional Euclidean space V is called an orthogonal basis, if it consists of nonzero vectors, which are pairwise orthogonal: $(e_i, e_j) = 0$ for $i \neq j$. An orthogonal basis is called orthonormal, if all its vectors are of length 1.

Lemma 1. An orthogonal basis is a basis.

Indeed, assuming $c_1e_1 + \ldots + c_ne_n = 0$, we have

$$0 = (0, e_k) = (c_1e_1+\ldots+c_ne_n, e_k) = c_1(e_1, e_k)+\ldots+c_n(e_n, e_k) = c_k(e_k, e_k),$$

which implies $c_k = 0$, since $e_k \neq 0$. (For any vector v we have $(0, v) = 0$ since $(0, v) = (2 \cdot 0, v) = 2(0, v)$.) Thus our system is linearly independent, and contains $\dim V$ vectors, so is a basis.

Lemma 2. Any n-dimensional Euclidean space contains orthogonal bases.

We shall start from any basis $f_1, \ldots, f_n$, and transform it into an orthogonal basis. Namely, we shall prove by induction that there exists a basis $e_1, \ldots, e_k, f_{k+1}, \ldots, f_n$, where the first k vectors are pairwise orthogonal. Induction base is trivial, as for $k = 1$ there are no pairwise distinct vectors to be orthogonal, and we can put $e_1 = f_1$. Assume that our statement is proved for some k, and let us show how to deduce it for $k+1$. Let us search for e_{k+1} of the form $f_{k+1} - a_1e_1 - \ldots - a_ke_k$. Conditions $(e_{k+1}, e_j) = 0$ for $j = 1, \ldots, k$ mean that

$$0 = (f_{k+1} - a_1e_1 - \ldots - a_ke_k, e_j) = (f_{k+1}, e_j) - a_1(e_1, e_j) - \ldots - a_k(e_k, e_j),$$

and the induction hypothesis guarantees that the latter is equal to

$$(f_{k+1}, e_j) - a_j(e_j, e_j),$$

so we can put $a_j = \frac{(f_{k+1}, e_j)}{(e_j, e_j)}$. Let us show that the vector thus obtained is nonzero. From the very nature of our procedure, e_2 is a linear combination of f_1 and f_2, ..., e_k is a linear combination of $f_1, \ldots, f_k$, so $a_1e_1 + \ldots + a_ke_k$ is a linear combination of $f_1, \ldots, f_k$, and

$$f_{k+1} - a_1e_1 - \ldots - a_ke_k \neq 0$$

since $f_1, \ldots, f_n$ form a basis. This completes the proof of the induction step.

The procedure described above is called *Gram-Schmidt orthogonalisation procedure.* If after orthogonalisation we divide all vectors by their lengths, we obtain an orthonormal basis.

Lemma 3. For any inner product and any basis $e_1, \dots, e_n$ of V, we have

$$(x_1e_1 + \ldots + x_ne_n, y_1e_1 + \ldots + y_ne_n) = \sum_{i,j=1}^{n} a_{ij}x_iy_j,$$

where $a_{ij} = (e_i, e_j)$.

This follows immediately from linearity property of inner products.

Corollary. A basis $e_1, \dots, e_n$ is orthonormal if and only if

$$(x_1e_1 + \ldots + x_ne_n, y_1e_1 + \ldots + y_ne_n) = x_1y_1 + \ldots + x_ny_n.$$

Corollary. A basis $e_1, \dots, e_n$ is orthonormal if and only if for any vector v its k^{th} coordinate is equal to (v, e_k):

$$v = (v, e_1)e_1 + \ldots + (v, e_n)e_n.$$

Lemma 4. Any orthonormal system of vectors in an n-dimensional Euclidean space can be included in an orthonormal basis.

Indeed, a reasoning similar to the one given above would show that this system is linearly independent. Thus it can be extended to a basis. If we apply the orthogonalisation procedure to this basis, we shall end up with an orthonormal basis containing our system (nothing would happen to our vectors during orthogonalisation).

Definition 1. Let U be a subspace of a Euclidean space V. The set of all vectors v such that $(v, u) = 0$ for all $u \in U$ is called the orthogonal complement of U, and is denoted by $U^\perp$.

Lemma 5. For any subspace U, $U^\perp$ is also a subspace.

This follows immediately from linearity property of inner products.

Lemma 6. For any subspace U, we have $U \cap U^\perp = \{0\}$.

Indeed, if $u \in U \cap U^\perp$, we have $(u, u) = 0$, so $u = 0$.

Lemma 7. For any finite-dimensional subspace $U \subset V$, we have $V = U \oplus U^\perp$. (This justifies the name "orthogonal complement" for $U^\perp$.)

(In the lecture, that was proved for a finite-dimensional V, but here we shall prove it for a more general case, where we have no assumptions on V.)

Let $e_1, \dots, e_k$ be an orthonormal basis of U. To prove that the direct sum coincides with V, it is enough to prove that any vector $v \in V$ can be represented in the form $u + u^\perp$, where $u \in U$, $u^\perp \in U^\perp$, or, equivalently, in the form $c_1e_1 + \ldots + c_ke_k + u^\perp$, where $c_1, \dots, c_k$ are unknown coefficients. Computing inner products with e_j for $j = 1, \dots, k$, we get a system of equations to determine c_i:

$$(c_1e_1 + \ldots + c_ke_k + u^\perp, e_j) = (v, e_j).$$

Due to orthonormality of our basis and the definition of the orthogonal complement, the left hand side of this equation is c_j. On the other hand, it is easy to see that for any v, the vector

$$v - (v, e_1)e_1 - \ldots, (v, e_k)e_k$$

is orthogonal to all e_j, and so to all vectors from U, and so belongs to $U^\perp$. The lemma is proved.

Definition 2. In the notation of the previous proof, u is called the projection of v onto U and $u^\perp$ is called the perpendicular dropped from v on U.

Lemma 8. $|u^\perp|$ is the shortest distance from the endpoint of v to points of U:

$$|u^\perp| \geqslant |v - u_1|$$

for any $u_1 \in U$.

Indeed, $|v - u_1|^2 = |v - u + u - u_1|^2 = |v - u|^2 + |u - u_1|^2$ due to the Pythagoras theorem, so $|v - u_1|^2 \geqslant |v - u|^2$.

Corollary (Bessel's inequality). For any vector $v \in V$ and any orthonormal system e_1, $\ldots$, e_k (not necessarily a basis) we have

$$(v, v) \geqslant (v, e_1)^2 + \ldots + (v, e_k)^2.$$

Indeed, we can take $U = \mathrm{span}(e_1, \ldots, e_k)$ and represent $v = u + u^\perp$. Then

$$|v|^2 = |u|^2 + |u^\perp|^2 \geqslant |u|^2 = (u, e_1)^2 + \ldots + (u, e_k)^2 = (v, e_1)^2 + \ldots + (v, e_k)^2.$$

Example 1. Consider the Eucludean space of all continuous functions on $[-\pi, \pi]$ with an inner product

$$(f(t), g(t)) = \int_{-\pi}^{\pi} f(t)g(t)\,dt.$$

It is easy to see that the functions

$$e_0 = \frac{1}{\sqrt{2\pi}}, e_1 = \frac{\cos t}{\sqrt{\pi}}, f_1 = \frac{\sin t}{\sqrt{\pi}}, \ldots, e_n = \frac{\cos nt}{\sqrt{\pi}}, f_n = \frac{\sin nt}{\sqrt{\pi}}$$

form an orthonormal system there. Consider the function $h(t) = t$. We have

$$(h(t), h(t)) = \frac{2\pi^3}{3},$$
$$(h(t), e_0) = 0),$$
$$(h(t), e_k) = 0,$$
$$(h(t), f_k) = \frac{2(-1)^{k+1}\sqrt{\pi}}{k},$$

(the latter integral requires integration by parts to compute it), so Bessel's inequality implies that

$$\frac{2\pi^3}{3} \geqslant 4\pi + \frac{4\pi}{4} + \frac{4\pi}{9} + \ldots + \frac{4\pi}{n^2},$$

which can be rewritten as

$$\frac{\pi^2}{6} \geqslant 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2}.$$

Actually $\sum_k \frac{1}{k^2} = \frac{\pi^2}{6}$, which was first proved by Euler. We are not able to establish it here, but it is worth mentioning that Bessel's inequality gives a sharp bound for this sum.