

Solutions to the Michaelmas term paper

1. There are many ways to solve this problem. Let us sketch some of them. First of all, let us remark that for a vector perpendicular to the first plane we can take

$$((5, 1, -3) - (0, 1, -1)) \times ((6, -3, 3) - (5, 1, -3)) = (-8, -32, -20),$$

and for a vector perpendicular to the second plane we can take

$$((5, 1, -3) - (2, 1, -1)) \times ((5, 1, -3) - (1, -2, 3)) = (6, 10, 9).$$

A very direct way: similar to the one discussed in the solutions to the sample paper (compute the vectors perpendicular to our planes (as above), determine the equations of our planes, solve the corresponding system of equations, extract from the solution a vector going along the intersection line, and check whether or not it is proportional to \mathbf{n}).

A slightly more elegant way: the intersection line is perpendicular to both of the vectors perpendicular to our planes, so we can compute dot products of \mathbf{n} with these vectors and see whether they are equal to zero. For the first vector, we see that the dot product is $(-8) \cdot 2 + (-32) \cdot 4 + (-20) \cdot 1 \neq 0$, so these vectors are not perpendicular. This means that \mathbf{n} is not parallel to the intersection line.

Another slightly more elegant way: the intersection line, being perpendicular to both of the vectors perpendicular to our planes, goes along their cross product. The cross product of our vectors is

$$(-88, -48, 112),$$

which is not parallel to \mathbf{n} .

2. (a) $(A + B)(A - B) = A^2 - AB + BA - B^2$, so $(A + B)(A - B) = A^2 - B^2$ if and only if $AB = BA$.
- (b) That is not true. Let us give some counterexamples.

Example 1: we know that $(-1)^2 = 1$, so for each matrix that has ± 1 's on the diagonal and zeros outside the diagonal, its square is equal to I . For example, take $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Example 2: in class, we checked that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = 0$. But $0 = 0^2$ as well, and we have another counterexample.

Example 3: this was discovered independently by several students during the exam. For each a, b we have

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^2 = \begin{pmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} b & a \\ a & b \end{pmatrix}^2.$$

3. (a)

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -3 & 5 \end{pmatrix}, \quad A_+ = \begin{pmatrix} 2 & 1 & -1 & 4 \\ 1 & 0 & 1 & 3 \\ 1 & -3 & 5 & 1 \end{pmatrix},$$

Expanding along the second row, we have $\det(A) = -2 - (-7) = 5$. Since $\det(A) \neq 0$, we see that A is invertible.

(b)

$$A_+ \xrightarrow{(1) \leftrightarrow (2)} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 2 & 1 & -1 & 4 \\ 1 & -3 & 5 & 1 \end{pmatrix} \xrightarrow{(2)-2(1), (3)-(1)} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 4 & -2 \end{pmatrix} \xrightarrow{(3)+3(2)} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -5 & -8 \end{pmatrix} \xrightarrow{\frac{1}{5} \cdot (3), (2)+3(3), (1)-(3)} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{5} \\ 0 & 1 & 0 & \frac{14}{5} \\ 0 & 0 & 1 & \frac{8}{5} \end{pmatrix}.$$

This means that the system

$$\begin{cases} x_1 = \frac{7}{5}, \\ x_2 = \frac{14}{5}, \\ x_3 = \frac{8}{5} \end{cases}$$

is equivalent to the original system. This gives us the only solution.

(c) Cramer's rule tells us that the only solution to our system is

$$\begin{cases} x_1 = \frac{\det(A_1)}{\det(A)}, \\ x_2 = \frac{\det(A_2)}{\det(A)}, \\ x_3 = \frac{\det(A_3)}{\det(A)}, \end{cases}$$

where

$$A_1 = \begin{pmatrix} 4 & 1 & -1 \\ 3 & 0 & 1 \\ 1 & -3 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 4 & -1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \\ 1 & -3 & 1 \end{pmatrix}.$$

Computing these determinants, we obtain the solution

$$\begin{cases} x_1 = \frac{7}{5}, \\ x_2 = \frac{14}{5}, \\ x_3 = \frac{8}{5}. \end{cases}$$

4. Since $(i, 2, k, 7, 4, 6, 1)$ should form a permutation, we have either $i = 3, k = 5$ or $i = 5, k = 3$. Similarly, since $(1, 4, 5, 3, 2, j, l)$ should form a permutation, we have either $j = 6, l = 7$ or $j = 7, l = 6$. The permutation $(3, 2, 5, 7, 4, 6, 1)$ has 10 inversions, the permutation $(5, 2, 3, 7, 4, 6, 1)$ has 11 inversions, the permutation $(1, 4, 5, 3, 2, 6, 7)$ has 5 inversions, and the permutation $(1, 4, 5, 3, 2, 7, 6)$ has 6 inversions. In class, we proved that to compute the sign of a summand in the determinant, we should compute the total number of inversions in the permutation of its first indices and the permutation of its second indices. It follows that either $i = 3, j = 6, k = 5, l = 7$, or $i = 5, j = 7, k = 3, l = 6$.
5. A matrix is not invertible if and only if its determinant is equal to zero. In our case, the determinant is

$$-t^3 + 5t^2 - 2t - 2 = -(t - 1)(t^2 - 4t - 2).$$

This expression is equal to zero in the following cases: $t = 1$ or $t = 2 \pm \sqrt{6}$.