Solutions to the sample Christmas exam paper

1. First of all, let us determine the equation of the plane β. The vector connecting the first point to the third one has coordinates $(1, 0, 0)$, the vector connecting the third poind to the second one has coordinates $(2, 2, 1)$. Their cross product $(0, -1, 2)$ is perpendicular to our plane, so its equation should be $-y + 2z = D$ for some D, and substitution of either of the given points gives us $D = 4$. Thus, the intersection line of our planes coincides with the solution set to the system of linear equations

$$\begin{cases} 2x + 3y + z = -3, \\ \quad -y + 2z = 4. \end{cases}$$

Solving this system, we see that the solution set consists of all points of the form $(\frac{9-7t}{2}, 2t - 4, t)$, where t is a parameter, which can be represented in the form

$$(\frac{9}{2}, -4, 0) + t(-\frac{7}{2}, 2, 1),$$

so for a vector parallel to our line we can take $\mathbf{b} = (-\frac{7}{2}, 2, 1)$. Let us denote by φ the angle between this vector and $\mathbf{a} = (1, 0, 1)$. We have

$$\cos\varphi = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-5/2}{\sqrt{2}\sqrt{69/4}} = -\frac{5}{\sqrt{138}}.$$

2. (**a**) Clearly,

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 1 & -1 \\ 5 & -1 & 1 \end{pmatrix}.$$

Expanding along the first row, we get

$$\det(A) = \det\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - 4\det\begin{pmatrix} 1 & -1 \\ 5 & 1 \end{pmatrix} + 2\det\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix} = 0 - 24 - 12 = -36.$$

As a consequence, A is invertible, since $\det(A) \neq 0$.

(**b**)

$$\begin{pmatrix}1 & 4 & 2 & 1 & 0 & 0\\ 1 & 1 & -1 & 0 & 1 & 0\\ 5 & -1 & 1 & 0 & 0 & 1\end{pmatrix} \overset{(2)-(1),(3)-5(1)}{\longmapsto}$$

$$\begin{pmatrix}1 & 4 & 2 & 1 & 0 & 0\\ 0 & -3 & -3 & -1 & 1 & 0\\ 0 & -21 & -9 & -5 & 0 & 1\end{pmatrix} \overset{(3)-7(2),-1/3(2)}{\longmapsto}$$

$$\begin{pmatrix}1 & 4 & 2 & 1 & 0 & 0\\ 0 & 1 & 1 & 1/3 & -1/3 & 0\\ 0 & 0 & 12 & 2 & -7 & 1\end{pmatrix} \overset{(1)-4(2),1/12(3)}{\longmapsto}$$

$$\begin{pmatrix}1 & 0 & -2 & -1/3 & 4/3 & 0\\ 0 & 1 & 1 & 1/3 & -1/3 & 0\\ 0 & 0 & 1 & 1/6 & -7/12 & 1/12\end{pmatrix} \overset{(1)+2(3),(2)-(3)}{\longmapsto}$$

$$\begin{pmatrix}1 & 0 & 0 & 0 & 1/6 & 1/6\\ 0 & 1 & 0 & 1/6 & 1/4 & -1/12\\ 0 & 0 & 1 & 1/6 & -7/12 & 1/12\end{pmatrix},$$

so

$$A^{-1} = \begin{pmatrix}0 & 1/6 & 1/6\\ 1/6 & 1/4 & -1/12\\ 1/6 & -7/12 & 1/12\end{pmatrix}.$$

The only solution to the system $Ax = b$ is $x = A^{-1}b$, which in our case gives the vector $\begin{pmatrix}1\\ 2/3\\ -4/3\end{pmatrix}$.

(**c**) The adjoint matrix is

$$\begin{pmatrix}0 & -6 & -6\\ -6 & -9 & 3\\ -6 & 21 & -3\end{pmatrix},$$

so using the formula $A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$, we obtain the same result as in the previous question.

3. *First solution*: as we know from lectures, to determine whether a permutation is even or odd, one can count the total number of inversions in both rows (for a 2-row notation). In our case, there are 17 inversions in the first row, and 27 inversions in the second row, which adds up to 44, so our permutation is even.

Second solution: if we rearrange columns in such a way that the the numbers in the first row are ordered properly, we get

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 5 & 4 & 10 & 6 & 2 & 8 & 7 & 9 \end{pmatrix},$$

and the second row has 12 inversions, so the permutation is even.

4. *First solution*: If $Ax = 0$ has only the trivial solution, then, as we know from one of the theorems proved in class, A is invertible. Then A^4 is also invertible: $(A^4)^{-1} = (A^{-1})^4$, and the same theorem (where it is proved that nonexistence of nontrivial solutions is equivalent to invertibility) guarantees that $A^4x = 0$ has only the trivial solution. If $Ax = 0$ has a nontrivial solution, it will also be a nontrivial solution to $A^4x = 0$, since $A^4x = A^3 \cdot Ax$.

Second solution: If $Ax = 0$ has a nontrivial solution, then $A^4x = 0$ has the same nontrivial solution (see above). Assume that $A^4x = 0$ has a nontrivial solution u. Since $A^4u = A \cdot A^3u$, we either have $A^3u = u$, or A^3u is a nontrivial solution to $Ax = 0$. In the second case, we proved what we wanted to prove, in the first case we notice that $A^3u = A \cdot A^2u$, and again, either $A^2u = 0$ or A^2u is a nontrivial solution to $Ax = 0$. Finally, $A^2u = A \cdot Au$, so either $Au = 0$ (so u is a nontrivial solution to $Ax = 0$) or Au is a nontrivial solution to $Ax = 0$.

5. Expanding the determinant of A_n along the first row, we get

$$\det(A_n) = 2\det(A_{n-1} + (-1) \cdot (-1) \cdot \det \begin{pmatrix} -1 & -1 & 0 & \dots & 0 \\ 0 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & -1 & 2 \end{pmatrix},$$

and expanding this determinant along the first column, we get the required formula

$$\det(A_n) = 2\det(A_{n-1}) - \det(A_{n-2}).$$

Furthermore, it is clear that $\det(A_1) = 2$, $\det(A_2) = 3$. Using the formula we obtained, we see that $\det(A_3) = 4$ and $\det(A_4) = 5$, so it is natural to assume that $\det(A_n) = n+1$. This formula is easy to prove by induction. We already have the induction basis ($n = 1, 2$), and if we know that $\det(A_k) = k + 1$, $\det(A_{k+1}) = k + 2$, we see that $\det(A_{k+2}) = 2(k + 2) - (k + 1) = k + 3$, so the step of induction can be easily made (note that unlike the most frequent way of proving things by induction, that is moving from k to $k + 1$, we here move in a less trivial way, from k and $k + 1$ to $k + 1$ and $k + 2$; to make use of our formula, we need to use pairs of consecutive determinants.