Solutions to the Hilary term exam paper

1. (a) A system of vectors of a vector space V is said to be complete if every vector in V can be represented as a linear combination of these vectors.

   Assume that the vectors $v_1, \ldots, v_k$ form a complete system in V. Let us extend them by some vector v. To prove that the new system of vectors is complete, let us take a vector $w \in V$ and find a linear combination of $v_1, \ldots, v_k$ which is equal to w. This combination can be regarded as a combination of $v_1, \ldots, v_k$, v with the coefficient of v equal to zero. Thus, the extended system is also complete.

   (b) For each $x \in V$, we would like to find some coefficients a', b', c' such that

   $$x = a'\mathbf{u}' + b'\mathbf{v}' + c'\mathbf{w}' = (a' + b')\mathbf{u} + (a' + 2c')\mathbf{v} + (-b' + c')\mathbf{w}.$$

   Since the original system was complete, we can find some coefficients a, b, c for which

   $$x = a\mathbf{u} + b\mathbf{v} + c\mathbf{w},$$

   so it is enough to require $a = a' + b'$, $b = a' + 2c'$, $c = -b' + c'$ (many of the students stopped before that in their exam papers: they re-wrote the combination as a combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and decided that it was enough; however, we should show that EVERY linear combination can be obtained this way!). This system of equations with unknowns a', b' and c' has solutions for all a, b, c (for example, its matrix has a nonzero determinant), so it is possible to find a combination of our vectors equal to x, and the system of vectors is complete.

   In class, we proved that the number of elements in a linearly independent system of vectors is less than or equal to the number of vectors in a complete system, so the dimension of our vector space is at most three.

2. (a) For a linear operator A from a vector space V to the vector space W the rank of A, by definition, is equal to the dimension of the image of A.

   (b) To show that $A_{\mathbf{v}}$ is a linear operator, we should show that $A_{\mathbf{v}}(\mathbf{w}_1 + \mathbf{w}_2) = A_{\mathbf{v}}(\mathbf{w}_1) + A_{\mathbf{v}}(\mathbf{w}_2)$ and $A_{\mathbf{v}}(c\mathbf{w}) = cA_{\mathbf{v}}(\mathbf{w})$, or, in other words, that $\mathbf{v} \times (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \times \mathbf{w}_1 + \mathbf{v} \times \mathbf{w}_2$ and $\mathbf{v} \times (c\mathbf{w}) = c\mathbf{v} \times \mathbf{w}$; both these statements were proved in class. To compute the rank of our operator, we use the fact the the sum of the rank of an operator and the dimension of its kernel is the dimension of the source space. This means that for an operator from $\mathbb{R}^3$ to $\mathbb{R}^3$ its rank is equal to 2 if and only if its kernel is one-dimensional, so it is enough to prove that $\dim \operatorname{Ker} A_{\mathbf{v}} = 1$. The kernel of $A_{\mathbf{v}}$ consists of vectors whose cross product with $\mathbf{v}$ is equal to zero. Such a vector is either equal to zero or proportional to $\mathbf{v}$ (the length of the cross product is equal to the product of lengths times the sine of the angle...), which gives a one-dimensional space.

   (c) First of all, $\operatorname{Im}(AB) = \{ABu \colon u \in U\} \subset \{Av \colon v \in V\}$, so $\dim \operatorname{Im}(AB) \leq \dim \operatorname{Im}(A)$, $\operatorname{rk}(AB) \leq \operatorname{rk}(A)$. The same would not work for $\operatorname{Im}(B)$, since

$\mathrm{Im}(AB) \not\subset \mathrm{Im}(B)$, they are subspaces in two different spaces! However, $\mathrm{Ker}(B) \subset \mathrm{Ker}(AB)$ (if $Bx = 0$, then $ABx = A0 = 0$), so $\dim \mathrm{Ker}(B) \leq \dim \mathrm{Ker}(AB)$, and so

$$\mathrm{rk}(AB) = \dim U - \dim \mathrm{Ker}(AB) \leq \dim U - \dim \mathrm{Ker}(B) = \mathrm{rk}(B).$$

3. Consider the matrices

$$A = \begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) The characteristic polynomial of A is $-t^3 + t^2 + t - 1 = (1 - t^2)(t - 1)$, so the eigenvalues are 1 and -1. The characteristic polynomial of B is $(1-t)^3$, so all eigenvalues are equal to 1. Solving the corresponding linear systems, we see that for A every eigenvector with the eigenvalue 1 is proportional to $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and every eigenvector with the eigenvalue -1 is proportional to $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, and for B every eigenvector is of the form $\begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$.

(b) For the eigenvalue -1 of A, let us consider the matrix

$$C = A + I = \begin{pmatrix} 10 & 5 & 2 \\ -16 & -8 & -4 \\ 2 & 1 & 2 \end{pmatrix}.$$

This matrix has rank 2, and we have $A^2 = \begin{pmatrix} 24 & 12 & 4 \\ -40 & -20 & -8 \\ 8 & 4 & 4 \end{pmatrix}$, which also has rank 2. This means that for the eigenvalue -1 the sequence of kernels of powers of C stabilizes at the first step, and the only contribution to the Jordan basis is the basis of $\mathrm{Ker}(A + I)$, that is eigenvectors. We already know that for this basis we can take $e = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

For the eigenvalue 1 of A, let us consider the matrix

$$D = A - I = \begin{pmatrix} 8 & 5 & 2 \\ -16 & -10 & -4 \\ 2 & 1 & 0 \end{pmatrix}.$$

This matrix has rank 2, and we have $A^2 = \begin{pmatrix} -12 & -8 & -4 \\ 24 & 16 & 8 \\ 0 & 0 & 0 \end{pmatrix}$, which is the matrix of rank 1. This means that the eigenvalue 1 contributes at least a thread of length 2. But we know that our space is 3-dimensional, and we already have one basis vector for the other eigenvalue, so we have one thread of length 2, and the sequence of kernels of powers of D stabilizes at the second step. To describe the corresponding thread, we should find a basis of $\operatorname{Ker}(A-I)^2$ relative to $\operatorname{Ker}(A-I)$, that is a vector in $\operatorname{Ker}(A-I)^2$ which is not in the kernel of $A-I$. The kernel of $(A-I)^2$ consists of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with $-12x-8y-4z=0$. Let us take $x=1$, $y=0$, then $z=-3$. For the vector $f=\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$, we have $(A-I)f=\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}\ne 0$. Overall, the vectors $(A-I)f$, f, e form a Jordan basis of A, and the Jordan normal form of A is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(c) No, A is not similar to B: they have different sets of eigenvalues, different traces, different determinants...

(d) We have $A^n=CJ^nC^{-1}$, where C is the transition matrix to our Jordan basis, $C=\begin{pmatrix} 2 & 1 & -1 \\ -4 & 0 & 2 \\ 2 & -3 & 0 \end{pmatrix}$. It is easy to see that $C^{-1}=\begin{pmatrix} 3/2 & 3/4 & 1/2 \\ 1 & 1/2 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. Also, $J^n=\begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Substituting and computing the matrix products, we get

$$A^n=\begin{pmatrix} 2n+4-3(-1)^n & n+2-2(-1)^n & 1-(-1)^n \\ -4n-6+6(-1)^n & -2n-3+4(-1)^n & 2(-1)^n-2 \\ 2n & n & 1 \end{pmatrix}.$$

4. First of all, let us notice that all eigenvalues of A are equal to $\pm i$: if $Ax=\lambda x$ for some $x\ne 0$, we have $\lambda^2x=A^2x=-Ex=-x$, so $\lambda^2=-1$. Secondly, the trace of a matrix is equal to the sum of all eigenvalues (with multiplicities), so the trace of our matrix is a multiple of i. But our matrix has real matrix elements, so its trace is real, which implies that it should be equal to zero.