UNIVERSITY OF DUBLIN

XMA1212

TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Mathematics
JF Theoretical Physics
JF Two Subject Mod

Trinity Term 2011

Course 1212

Dr. Vladimir Dotsenko

ATTEMPT **ALL** QUESTIONS

For each task, the number of points you can get for a complete solution of that task is printed next to it.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that if A is invertible, then the reduced row echelon form of A is the identity matrix".

All vector spaces unless otherwise specified are over complex numbers.

Non-programmable calculators are permitted for this examination.

1. (25 points) Let V be a vector space. Show that for every three linear operators $A, B, C\colon V \to V$ we have

$$\mathrm{rk}(ABC) \leq \mathrm{rk}(B).$$

Show that if A and C are invertible, then $\mathrm{rk}(ABC) = \mathrm{rk}(B)$, and give an example showing that this equality might hold even if A or C is not invertible.

2. (a) (15 points) Determine the Jordan normal form and find some Jordan basis for the matrix

$$A = \begin{pmatrix} 2 & -5 & 3 \\ 2 & -6 & 4 \\ 3 & -9 & 6 \end{pmatrix}.$$

(b) (15 points) Find a closed formula for A^n.

3. (a) (5 points) Write down the definition of a bilinear form on a real vector space. Which symmetric bilinear forms are said to be positive definite?

(b) (15 points) A quadratic form Q on the three-dimensional space with a basis e_1, e_2, e_3 is defined by the formula

$$Q(xe_1 + ye_2 + ze_3) = 3x^2 + 2axy + (2 - 2a)xz + (a + 2)y^2 + 2ayz + 3z^2$$

Find all values of the parameter a for which this form is positive definite.

4. A square matrix A (of some size $n \times n$) satisfies the condition $A^2 - 8A + 15I = 0$.

(a) (15 points) Show that this matrix is similar to a diagonal matrix.

(b) (10 points) Show that for every positive integer $k \geq 8$ there exists a matrix A satisfying the above condition with $\mathrm{tr}(A) = k$.