Orthonormal bases, orthogonal complements, and orthogonal direct sums

A system of vectors $e_1, \dots, e_k$ of a Euclidean space V is said to be orthogonal, if it consists of nonzero vectors, which are pairwise orthogonal: $(e_i, e_j) = 0$ for $i \neq j$. An orthogonal system is said to be orthonormal, if all its vectors are of length 1: $(e_i, e_i) = 1$.

Lemma 1. An orthogonal system is linearly independent.

Indeed, assuming $c_1e_1 + \ldots + c_ke_k = 0$, we have

$$0 = (0, e_p) = (c_1e_1 + \ldots + c_ke_k, e_p) = c_1(e_1, e_p) + \ldots + c_k(e_k, e_p) = c_p(e_p, e_p),$$

which implies $c_p = 0$, since $e_p \neq 0$. (By the way, we have $(0, v) = 0$ for every vector v since $(0, v) = (2 \cdot 0, v) = 2(0, v)$.) Thus our system is linearly independent.

Lemma 2. Every finite-dimensional Euclidean space contains orthonormal bases.

We shall start from some basis f_1, ..., f_n, and transform it into an orthonormal basis. Namely, we shall prove by induction that there exists a basis e_1, ..., e_{k-1}, f_k, ..., f_n, where the first $(k-1)$ vectors form an orthonormal system and are equal to linear combinations of the first $(k-1)$ vectors of the original basis. Induction base is an empty statement and is trivial. Assume that our statement is proved for some k, and let us show how to deduce it for $k+1$. Let us search for e_k of the form $f_k - a_1e_1 - \ldots - a_{k-1}e_{k-1}$; this way the condition on linear combinations on the first k vectors of the original basis is automatically satisfied. Conditions $(e_k, e_j) = 0$ for $j = 1, \ldots, k-1$ mean that

$$0 = (f_k - a_1e_1 - \ldots - a_{k-1}e_{k-1}, e_j) = (f_k, e_j) - a_1(e_1, e_j) - \ldots - a_{k-1}(e_{k-1}, e_j),$$

and the induction hypothesis guarantees that the latter is equal to

$$(f_k, e_j) - a_j(e_j, e_j) = (f_k, e_j) - a_j,$$

so we can put $a_j = (f_k, e_j)$ for all $j = 1, \ldots, k-1$. Let us show that the vector thus obtained is nonzero. By the induction hypothesis, $a_1e_1 + \ldots + a_{k-1}e_{k-1}$ is a linear combination of f_1, ..., f_{k-1}, so in the linear combination

$$f_k - a_1e_1 - \ldots - a_{k-1}e_{k-1},$$

expressed as a combination of $f_1, \ldots, f_k$, the vector f_k appears with coefficient 1 and hence this combination is nonzero since f_1, ..., f_n form a basis.

To complete the proof of the induction step, we normalise the vector e_k, replacing it by $\frac{1}{\sqrt{(e_k,e_k)}}e_k$.

The procedure described above is called *Gram-Schmidt orthogonalisation procedure.*

Lemma 3. For every scalar product and every basis $e_1, \ldots, e_n$ of V, we have

$$(x_1e_1 + \ldots + x_ne_n, y_1e_1 + \ldots + y_ne_n) = \sum_{i,j=1}^{n} a_{ij}x_iy_j,$$

where $a_{ij} = (e_i, e_j)$.

This follows immediately from the bilinearity property of scalar products.

Corollary. A basis $e_1, \ldots, e_n$ is orthonormal if and only if

$$(x_1e_1 + \ldots + x_ne_n, y_1e_1 + \ldots + y_ne_n) = x_1y_1 + \ldots + x_ny_n.$$

In other words, an orthonormal basis provides us with a system of coordinates that identifies V with $\mathbb{R}^n$ with the standard scalar product.

Corollary. A basis $e_1, \ldots, e_n$ is orthonormal if and only if for every vector v its k^{th} coordinate is equal to (v, e_k):

$$v = (v, e_1)e_1 + \ldots + (v, e_n)e_n.$$

Lemma 4. Every orthonormal system of vectors in an n-dimensional Euclidean space can be included in an orthonormal basis.

Indeed, a reasoning similar to the one given above would show that this system is linearly independent. Thus it can be extended to a basis. If we apply the orthogonalisation procedure to this basis, we shall end up with an orthonormal basis containing our system (nothing would happen to our vectors during orthogonalisation).

Definition 1. Let U be a subspace of a Euclidean space V. The set of all vectors v such that $(v, u) = 0$ for all $u \in U$ is called the orthogonal complement of U, and is denoted by $U^\perp$.

Lemma 5. For every subspace U, $U^\perp$ is also a subspace.

This follows immediately from the bilinearity property of inner products.

Lemma 6. For every subspace U, we have $U \cap U^\perp = \{0\}$.

Indeed, if $u \in U \cap U^\perp$, we have $(u, u) = 0$, so $u = 0$.

Lemma 7. For every finite-dimensional subspace $U \subset V$, we have $V = U \oplus U^\perp$. (This justifies the name “orthogonal complement” for $U^\perp$.)

Let $e_1, \ldots, e_k$ be an orthonormal basis of U. To prove that the direct sum coincides with V, it is enough to prove that every vector $v \in V$ can be represented in the form $u + u^\perp$, where $u \in U$, $u^\perp \in U^\perp$, or, equivalently, in the form $c_1e_1 + \ldots + c_ke_k + u^\perp$, where $c_1, \ldots, c_k$ are unknown coefficients. Computing inner products with e_j for $j = 1, \ldots, k$, we get a system of equations to determine c_i:

$$(c_1e_1 + \ldots + c_ke_k + u^\perp, e_j) = (v, e_j).$$

Due to orthonormality of our basis and the definition of the orthogonal complement, the left hand side of this equation is c_j. On the other hand, it is easy to see that for every v, the vector

$$v - (v, e_1)e_1 - \ldots, (v, e_k)e_k$$

is orthogonal to all e_j, and so to all vectors from U, and so belongs to $U^\perp$. The lemma is proved.

Corollary (Bessel's inequality). For any vector $v \in V$ and any orthonormal system $e_1, \ldots, e_k$ (not necessarily a basis) we have

$$(v, v) \geqslant (v, e_1)^2 + \ldots + (v, e_k)^2.$$

Indeed, we can take $U = \text{span}(e_1, \ldots, e_k)$ and represent $v = u + u^\perp$. Then

$$|v|^2 = |u|^2 + |u^\perp|^2 \geqslant |u|^2 = (u, e_1)^2 + \ldots + (u, e_k)^2 = (v, e_1)^2 + \ldots + (v, e_k)^2.$$

Example 1. Consider the Eucludean space of all continuous functions on $[-1, 1]$ with the inner product

$$(f(t), g(t)) = \int_{-1}^{1} f(t)g(t)\,dt.$$

It is easy to see that the functions

$$e_0 = \frac{1}{\sqrt{2}}, e_1 = \cos \pi t, f_1 = \sin \pi t, \ldots, e_n = \cos \pi n t, f_n = \sin \pi n t$$

form an orthonormal system there. Consider the function $h(t) = t$. We have

$$(h(t), h(t)) = \frac{2}{3},$$
$$(h(t), e_0) = 0),$$
$$(h(t), e_k) = 0,$$
$$(h(t), f_k) = \frac{2(-1)^{k+1}}{k\pi},$$

(the latter integral requires integration by parts to compute it), so Bessel's inequality implies that

$$\frac{2}{3} \geqslant \frac{4}{\pi^2} + \frac{4}{4\pi^2} + \frac{4}{9\pi^2} + \ldots + \frac{4}{n^2\pi^2},$$

which can be rewritten as

$$\frac{\pi^2}{6} \geqslant 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2}.$$

Actually $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, which was first proved by Euler. We are not able to establish it here, but it is worth mentioning that Bessel's inequality gives a sharp bound for this sum.